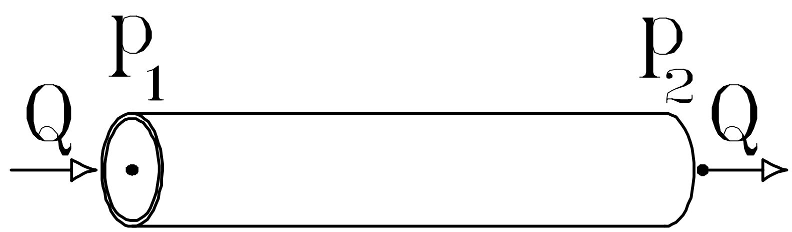
Problem Set No. 4
Due Friday, February 16, 2001

Fluid Systems

Mechanical engineers have an intuitive understanding of the basics of fluid systems because water flow is so familiar. We know that water flows in the direction of the pressure drop in the pipe. Knowing the pressure at one location (node) is insufficient information. We must know the pressure at two nodes and calculate the difference to know the flow direction.



The pressure drop in the direction of flow, p_{12}, is the **ACROSS VARIABLE** in fluid systems. We will measure flow as a Volume Flow Rate, Q. Q is the **THROUGH VARIABLE**; it flows through the pipe. Power in a fluid system is:

$$\mathbf{P} = p_{12}Q$$

where p_{12} is the pressure drop in the direction of the flow and Q is the volume flow rate.

We will work in SI units, so:

$$[Q] = \left[\frac{m^3}{sec}\right]$$

In US Customary Units, volume flow rate is often reported in units of gal/min, abbreviated GPM. If the power of a hydraulic pump is reported in GPM, then one must assume a hydraulic pressure, 2500 psi is a typical pressure, to calculate the power. Note the mix of units in US Customary usage. One would have to convert gallons to cubic feet and psi to psf in order to calculate power in units of foot-pounds per minute. The calculation is much more direct in SI units:

$$[\mathbf{P}] = [pQ] = \left[\frac{N}{m^2}\right]\left[\frac{m^3}{sec}\right] = \left[\frac{N-m}{sec}\right] = \left[\frac{Joule}{sec}\right] = Watt$$

Fluid systems are completely analogous to low frequency electrical systems. In fact, most mechanical engineers understand low frequency electrical systems by the fluid analogies. (The analogies break down when the frequency of an electrical system is high enough that the electrical energy must be modeled as transmitted by a wave rather than by particles.) For the electrical systems that mechanical engineers work with, fluid pressure is analogous to electrical voltage and fluid volume flow rate is analogous to electrical current. The only significant difference between network representations of electrical and fluid systems is that fluid capacitors must be referenced to atmospheric pressure, or some other "ground" pressure, whereas electrical capacitors do not.

Mechanical engineers work with many different types of fluid systems. Our emphasis will be on fluid power systems, either pneumatic or hydraulic, but our analyses can be applied to other types of fluid systems, such as water distribution systems.

Elemental and Energy Equations

Fluid systems are mechanical systems since they obey Newton's laws. They store energy as kinetic energy (energy of motion) or strain energy (energy of elastic deformation). Kinetic energy is dissipated as heat through viscous friction, which is lost from the system. One difference between fluid systems and our treatment of translational mechanical systems is that we will include gravity as a form potential energy storage in fluid systems, rather than as a force source.

Energy Dissipation: Fluid Resistance

Energy is dissipated as heat in fluid flow by viscous shear in components such as filters, orifices, and valves, due to changes in diameters and direction of pipes and the roughness of their inner surface. In the simplest case, the pressure drop in the direction of the flow is proportional to the flow rate:

$$p_{12} = RQ_R$$

This relationship has the same form as electrical resistance:

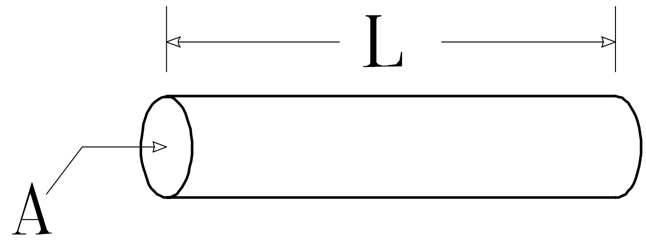
$$v_{12} = Ri_R$$

Unfortunately, the simplest case does not describe many practical situations. It is an accurate model for slow flows, such as through a filter or of a very viscous fluid. It looses accuracy as the fluid flow increases. We will ignore that unpleasant fact in ME 352.

Kinetic Energy Storage: Fluid Inertance

Fluid has mass. Consequently, flowing fluid possesses kinetic energy. We will derive the Elemental and Energy equations for the dynamic element which represents mass in a fluid system by expressing the corresponding equations for a translational mechanical system in terms of the fluid power variables p and Q.

Consider the mass of fluid with density ρ in a pipe of length L and cross-sectional area A, as shown below:



Cylinderical Fluid Mass

This slug of fluid has mass:

$$M = \rho AL$$

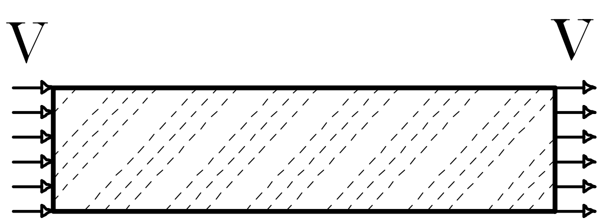
The mass must be acted on by a force if it is to accelerate or decelerate.

$$F = Ma = M\frac{dv}{dt} = \rho AL\frac{dv}{dt}$$

The force acting to accelerate the mass of fluid is the net pressure force between node 1 and node 2:

$$F_{\substack{Net\\Pressure}} = A(p_1 - p_2) = Ap_{12}$$

The actual velocity of fluid flowing down a pipe varies with radial and longitudinal position and depends on the fluid properties, the velocity of the flow, and geometry and surface roughness of the pipe. We will make the simplifying assumption that the velocity is uniform across the pipe, allowing us to visualize the fluid flowing down the pipe as a cylindrical slug. The assumption of uniform velocity is actually reasonable for turbulent flow, such as that of fluid flowing at typical velocities through components with the complicated geometries of hydraulic systems.



Volume of Fluid Moving with a Uniform Velocity Profile

We need to relate our power variable, Q, volume flow rate, to the assumed uniform velocity of the flow. The volume of fluid flowing at velocity v that passes through a plane normal to the flow of area A in a period of time t is:

$$\text{Volume} = Avt$$

Hence, the volume flow rate Q is:

$$\frac{\text{Volume}}{t} = Q = Av$$

We can now express velocity v in terms of volume flow rate Q:

$$v = \frac{Q}{A}$$

Now, we will write F = ma in terms of p_{12} and Q:

$$F_{\substack{\text{Net}\\ \text{Pressure}}} = Ma$$

$$p_{12}A = M\frac{dv}{dt}$$

$$p_{12}A = \rho AL\frac{dv}{dt}$$

$$p_{12}A = \rho AL\frac{d\left(\frac{Q}{A}\right)}{dt}$$

$$p_{12} = \frac{\rho L}{A}\frac{dQ}{dt}$$

$$p_{12} = I\frac{dQ}{dt}$$

The quantity $I = \frac{\rho L}{A}$ is called Fluid Inertance. Note that there is an apparent contradiction in the definition. Shouldn't increasing the cross-sectional area of the pipe increase the inertia of the fluid moving down it by increasing the mass of fluid in a length of pipe? It does, because Fluid Inertance is not inertia. Fluid inertance was derived using volume flow rate Q in place of fluid velocity v. Increasing the cross-sectional area of a pipe results in a **decrease** in the fluid velocity for a given volume flow rate Q. The acceleration of the fluid is also decreased proportionally. Hence, the Fluid Inertance I is inversely proportional to the cross-sectional area A of the pipe.

Using the above expressions for the mass of fluid in a length of pipe and the uniform velocity in terms of Q, we can write the expression for the Kinetic Energy stored in a flowing fluid:

$$\mathbf{E}_{\text{Kinetic}} = \frac{1}{2}Mv^2$$

$$\mathbf{E}_{\text{Kinetic}} = \frac{1}{2}\rho AL\left(\frac{Q}{A}\right)^2$$

$$\mathbf{E}_{\text{Kinetic}} = \frac{1}{2}\rho AL\left(\frac{Q}{A}\right)^2$$

$$\mathbf{E}_{\text{Kinetic}} = \frac{1}{2}\frac{\rho A}{A}Q^2$$

$$\mathbf{E}_{I} = \frac{1}{2}IQ^2$$

Where, again, fluid inertance I plays the role of mass in the equation.

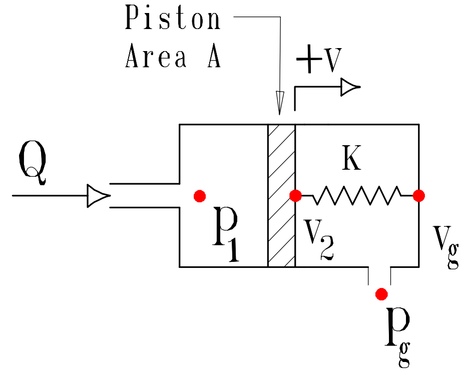
Pressure-Based Energy Storage: Elastic Strain Energy and Gravitational Potential Energy

Fluid systems can store energy in the elastic deformation of the pipe or structure containing the fluid under pressure, in the elastic deformation of the fluid itself, and by raising fluid against gravity. All three of these energy storage modes lead to elemental and energy equations with the same functional form as an electrical capacitor. The parameter used in these equations is given the analogous name, Fluid Capacitance, and the same symbol, C, as electrical capacitance.

In general, both the fluid and the components which contain the fluid are elastic and can store strain energy. These strain energies are calculated independently by first considering the fluid incompressible and calculating the strain energy stored in the components containing the fluid and then by considering those components to be rigid and calculating the strain energy stored in the fluid. Gravitational potential energy storage is usually insignificant in hydraulic systems in machines because the hydraulic pressures (on the order of 2500 psi) are so much larger than the gravitational pressures. Gravitational potential energy storage is important in water systems, both within a building and in a municipal distribution system.

Energy Stored in the Elastic Deformation of the Fluid Container

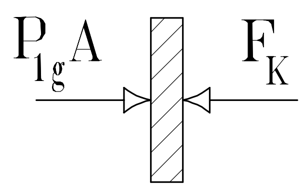
Consider a fluid system component which consists of a spring-loaded piston in a cylinder. This device is called a fluid "accumulator" in hydraulic systems, a "pressure relief tank" in hot water systems, and a fluid capacitor in system dynamics. Assume that the fluid is incompressible and the container is rigid, except for the spring. Note that the volume of the cylinder which contains the spring is vented to the reference (ground) pressure, usually atmospheric pressure. The only force acting on the right side of the piston is the spring force, F_K. Also note that there are two different types of across variable nodes identified on the schematic, pressure and velocity.



The volume flow rate Q is related to the velocity of the piston relative to ground, v_{2g}:

$$v_{2g} = \frac{Q}{A}$$

A free body diagram of the forces acting on the piston lead to an equilibrium equation which relates the net pressure force, $P_{1g}A$ of the fluid and the force in the spring, F_K:



The elemental equation for this fluid capacitor can be derived from the elemental equation for a spring:

$$\frac{dF_K}{dt} = Kv_{2g}$$

$$\frac{d\left(p_{1g}A\right)}{dt} = K\frac{Q}{A}$$

$$Q = \frac{A^2}{K}\frac{dp_{1g}}{dt}$$

This equation has the same form as the equation for electrical capacitance:

$$i = C\frac{dv}{dt}$$

Consequently, the capacitance of a fluid accumulator is:

$$C = \frac{A^2}{K}$$

The spring used in a fluid accumulator may a conventional compression spring. An alternative design is to use a diaphragm, rather than a piston, and high pressure gas, typically nitrogen, as the "spring". A component may act as an unintentional fluid capacitor if it exhibits sufficient elastic deformation when filled with a pressurized fluid. A common example is a garden hose. Many garden hoses expand significantly when pressurized, causing a delay in delivery of water from the hose. A significant amount of strain energy is stored in the hose even under the relatively low pressure of a residential water system of approximately 50 psi. The energy is released when the tap is closed, allowing water to continue to flow out of the hose.

The energy equation for this fluid capacitor can be derived from the energy storage equation for the spring:

$$\mathbf{E}_K = \frac{F_K^2}{2K}$$

$$\mathbf{E}_K = \frac{\left(A p_{1g}\right)^2}{2K}$$

$$\mathbf{E}_C = \frac{1}{2}\frac{A^2}{K} p_{1g}^2$$

$$\mathbf{E}_C = \frac{1}{2} C p_{1g}^2$$

Energy Stored in the Deformation of the Fluid

Positive stress is tensile in solids. The analog of stress in fluids in pressure. Positive pressure is compressive. Liquids in fluid power systems are kept under positive pressure to prevent "cavitation", or local vaporization, of the fluid. The problem is the damage to machinery from the stress created by the collapse of the bubbles when they are carried by the flow from a region of negative pressure into positive pressure. Bubbles and dissolved air in hydraulic oil are always problems because they greatly increase the compressibility of the fluid.

Strain in a fluid is defined as volumetric strain, a fractional change in volume. It is generally most convenient to express volumetric strain by its effect of increasing the density of the fluid as a fractional change in density:

$$\text{Volumetric Strain} \equiv \frac{d\rho}{\rho}$$

In the simplest case, there is a linear relationship between the an applied pressure and the volumetric strain:

$$dp = \beta \frac{d\rho}{\rho}$$

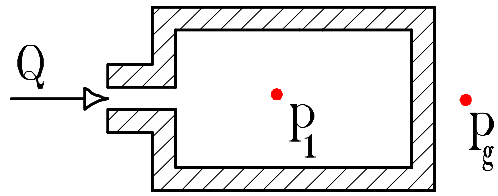
where β is the "Bulk Modulus". Note that this relationship has the same form as Hooke's Law for linear strain in a solid:

$$\sigma = E\varepsilon$$

The similarity is stronger than it may appear because Hooke's law can be written as:

$$d\sigma = E\frac{dL}{L}$$

The energy stored in the deformation of the fluid is calculated by assuming that the container is rigid and that more fluid is packed into the fixed volume $\mathbf{V}$ of the tank by compressing the fluid at increasing pressure.



Rigid Tank of Fixed Volume V

Conservation of mass requires that the mass of fluid that flows into the tank of fixed volume $\mathbf{V}$ must be stored in the tank by increasing the density of the fluid.

$$\text{Mass Flow Rate In} = \frac{d\left(\text{Mass}_{\text{Stored}}\right)}{dt}$$

$$\rho Q = \mathbf{V}\frac{d\rho}{dt}$$

Expressing dρ in terms of the bulk modulus β:

$$\rho Q = \mathbf{V}\frac{\rho}{\beta}\frac{dp_{1g}}{dt}$$

$$Q = \frac{\mathbf{V}}{\beta}\frac{dp_{1g}}{dt}$$

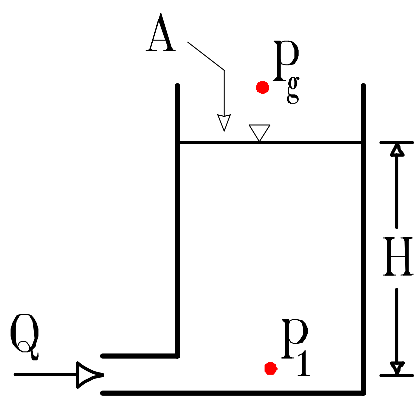
In this expression for fluid capacitance:

$$C = \frac{\mathbf{V}}{\beta}$$

Gravitational Potential Energy

Gravity was not included as an energy storage in translational mechanical systems because it is easier to include it as a force source which acts on mass. Gravity modelled as a force source still provides energy storage because power can flow both into and out of an ideal source. A car battery is a good example of a source which can either deliver (source) or accept (sink) power.

In fluid systems, it is easier to include gravity as a pressure-based energy storage, where the pressure is created by the height of a column of fluid. The most familiar example of a fluid capacitor of this type is a municipal water tower. If the water "tower" is a cylindrical tank, then it is called a "stand pipe":



The fluid capacitance is derived by assuming that the fluid is incompressible and the standpipe is rigid. The cross-sectional area of the standpipe is A. The density of the fluid is ρ. Conservation of mass dictates that the mass of fluid that flows into the port is stored in the standpipe. Expressing mass conservation in terms of mass flow rate:

$$\text{Mass Flow Rate In} = \frac{d\left(\text{Mass}_{\text{Stored}}\right)}{dt}$$

$$Q\rho = \rho A \frac{dH}{dt}$$

We can introduce our power variable pressure by recognizing that the pressure at the bottom of the tank relative to atmospheric (ground) pressure is the weight of a column of fluid of unit area, where g on the right side is the acceleration of gravity:

$$p_{1g} = \rho g H$$

We introduce g into right side of the mass conservation equation by multiplying and dividing by g:

$$Q\rho = \frac{g}{g}\rho A \frac{dH}{dt}$$

$$Q\rho = \frac{A}{g}\rho g \frac{dH}{dt}$$

$$Q = \frac{A}{\rho g}\frac{dp_{1g}}{dt}$$

In this form of fluid capacitance,

$$C = \frac{A}{\rho g}$$

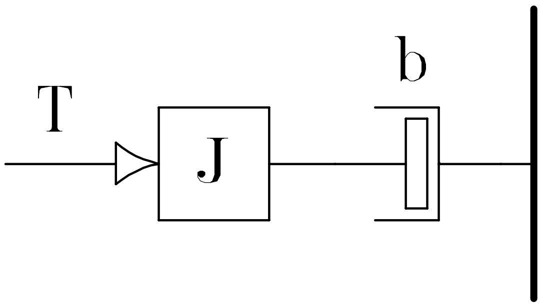
Drawing Linear Graphs from Fluid System Schematics

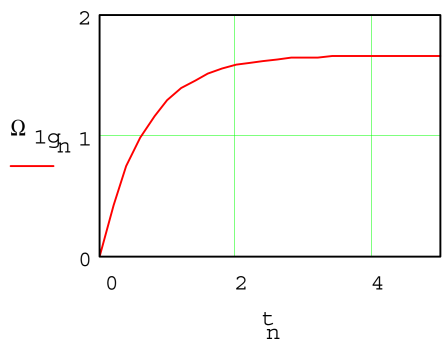
The reduction of fluid systems to a system equation follows the same procedure as for mechanical and electrical system. The first step is most important: **FIND THE NODES OF DISTINCT VALUES OF THE ACROSS VARIABLE PRESSURE.** Remember that there are always two nodes associated with each element. Always put a node at either end of pressure or flow sources, fluid resistors and fluid inertances. One node of a fluid capacitor is atmospheric pressure (ground). The other node is the maximum pressure in the fluid capacitor, at bottom of the tank.

Fluid system schematics contain ideal pipes which have no resistance, inertance, or capacitance. Ideal pipes are analogous to ideal conductors in electric circuits and massless and rigid rods or shafts in mechanical schematics. Do not assume properties for an ideal pipe. Only assign pressure nodes to elements that have been identified on the schematic as having energetic properties. The most common element to overlook is a fluid inertance. Don't forget that F = ma applies to fluids. There has to be a pressure drop to accelerate or decelerate fluid flow in a pipe. There are pressure nodes at either end of a fluid inertance.

Problem Set No. 4
Due Friday, February 16, 2001

Problem 1: A rotational mechanical system consisting of a torque source, a rotational inertia, and a rotational damper is shown below. A test was performed in which a step input torque of 5 N-m was applied to the system and the angular velocity of the inertia measured. The data, in units of seconds and radians per second, are plotted and tabulated below. Determine the magnitudes of the inertia J and the damping coefficient b. Interpolate if necessary.



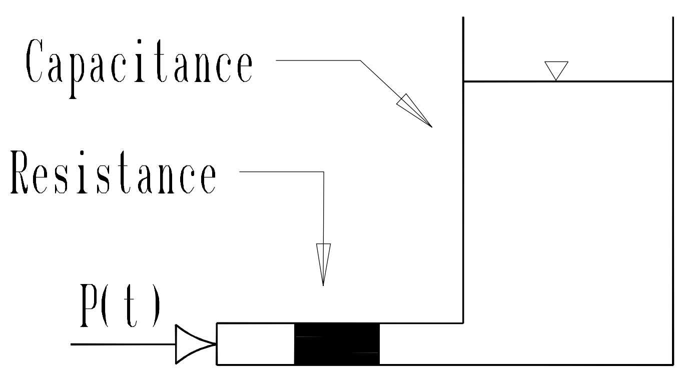


Time (sec)	Ω_{1g} (rad/sec)	Time (sec)	Ω_{1g} (rad/sec)
0	0	2.6	1.633
0.2	0.432	2.8	1.642
0.4	0.752	3.0	1.648
0.6	0.989	3.2	1.653
0.8	1.165	3.4	1.657
1.0	1.295	3.6	1.659
1.2	1.391	3.8	1.661
1.4	1.463	4.0	1.663
1.6	1.515	4.2	1.664
1.8	1.555	4.4	1.664
2.0	1.584	4.6	1.665
2.2	1.605	4.8	1.665
2.4	1.621	5.0	1.666

Problem 2: A fluid system consisting of a fluid resistance and fluid capacitance is acted upon by a pressure source. The capacitor is empty at time t = 0-. A step change in pressure P is applied at time t = 0. Derive and solve the system equations for:

a) The pressure in the fluid capacitor.

b) The volume flow rate through the fluid resistance.



Problem 3: A fluid system consisting of two fluid resistances and fluid capacitance is acted upon by a pressure source. The capacitor is empty at time t = 0-. A step change in pressure P is applied at time t = 0 to resistance R_1. Resistance R_2 drains to atmospheric pressure. Derive and solve the system equations for:

a) The volume flow rate through resistance R_1.

b) The volume flow rate through resistance R_2.