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Lafayette College

ME 352: Dynamics of Physical Systems and Electric Circuits

Problem Set No. 3
Due Monday, February 12, 2001

Reading: Rowell and Wormley, Chapters 3 and 4.

Linear Graphs and Mechanical Systems

We can use the techniques of circuit analysis to analyze any energetic system, if we can represent that system as a network of interacting energetic elements. The technique we will use is based on the physical and mathematical analogies between energetic systems. It will be convenient to define, some what arbitrarily, two general types of power variables to strengthen the analogies between different energetic systems.

Power in electrical, mechanical, and fluid systems is the product of two variables:

$$\mathbf{P} = iv = Fv = T\Omega = Qp$$

where i = current, F = force, T = torque, Q = volume flow rate, v = voltage or translational velocity, Ω = angular velocity, and p = fluid pressure. We will divide these variables into two groups based on how physical measurements of the variables are made.

Recall from Instrumentation that in order to measure the current flowing **THROUG**H an a circuit element, an ammeter needed to be inserted into a circuit in a series connection. In order to measure current i, it must flow **THROUG**H the ammeter. Similarly, to measure a force or a torque it must act **THROUGH** a load cell. Likewise, fluid must flow **THROUGH** a flow meter in order to measure the volume flow rate Q. We will call current, force, torque, and volume flow rate **THROUGH VARIABLES** and develop physical and mathematical analogies between them.

Continuity equations are summations of through variables at a node. Continuity equations in electrical and fluid statements are statements of conservation of charge and volume flow rate, respectively. What flows into a node must flow out, because a node has no physical properties. It can not store or dissipate the through variable. In translational and rotational mechanical systems, continuity equations are equilibrium statements. Forces or torques must sum to zero at a node. Note, because we are dealing with dynamic systems, we will include "inertial" forces and torques in our networks. (Inertial forces and torques are the F in F = ma and the T in $T = J\alpha$, the forces and torques which cause acceleration.) Consequently, forces and torques will sum to zero at nodes.

Recall from Instrumentation that the voltage driving a current flow through an element is the voltage drop **ACROSS** that element. Voltage drops are measured **ACROSS** an element by using the two probes of a voltmeter meter to create a parallel connection between the nodes at either end of the element. Voltage is a relative measurement. The voltage of a node cannot be defined without a reference. To have meaning, you must measure the voltage of a node relative to another node. In a mechanical system, the displacement of a single node of the spring is insufficient information to calculate the force or the strain energy stored in a spring. We need the deformation of the spring, which we measure deformation by measuring the difference in displacement between the two ends of the spring; in other words, **ACROSS** the spring. Likewise two nodes are needed to measure translational or rotational velocity. To calculate the kinetic energy of a mass or a rotational inertia, we need to measure the velocity of the mass or the inertia relative to an “inertial” (non-accelerating) reference. We will call an inertial reference in a mechanical system “ground”. The measurement of the velocity of a mass (or inertia) is **ACROSS** the mass (or inertia) to ground. Lastly, in a fluid system, fluid flow is driven by the pressure drop **ACROSS** an element, such as a valve. As with electrical voltage, it is not the magnitude of pressure that drives flow but the drop in pressure **ACROSS** an element. The pressure at one node is insufficient information. We will call voltage, translational velocity, rotational velocity, and pressure **ACROSS VARIABLES**.

Compatibility equations are summations of differences (or drops) in an across variable from node to node around a loop in a network. The differences from node to node sum to zero when you return to the node you started at. In electrical and fluid systems, the across variables voltage and pressure are “potentials”. These potentials express the net force that acts to drive the flow of charge or fluid. In translational and rotational mechanical systems, compatibility statements express geometric compatibility in terms of the derivative of displacement, velocity. If both ends of a spring have the same displacement, then there is no deformation of the spring. Similarly, if both ends of a spring have the same velocity, then the spring is not deforming. We use velocity instead of displacement so that we can work with differential equations rather than integral equations.

The analogies between through variables and across variable summarize as:

	Through Variable	**Across Variable**
Electrical	Current = i	Voltage = v
Translational Mechanical	Force = F	Velocity = v
Rotational Mechanical	Torque = T	Angular Velocity = Ω
Fluid	Volume Flow Rate = Q	Pressure = p

Mechanical Systems

Mechanical systems store energy as kinetic energy (energy of motion) or strain energy (energy of elastic deformation). Kinetic energy is dissipated as heat through friction, which is lost from the system. We will categorize mechanical systems based on the type of motion, either translation (linear) or rotation. Further, we will restricted motion in a system to one dimension, so that we can work with scalar velocities rather than vector velocities. It may be that we will need a number of sub-systems, each representing motion in one dimension, to represent the motion of a machine.

Elemental Equations

In a translational mechanical system,

$$\mathbf{P} = Fv$$

The most familiar mechanical element is a mass. Everyone's favorite equation from mechanics is:

$$F = ma$$

Recall that elemental equations relate the two power variables in the system. However, $F = ma$ is not written in terms of the power variables F and v. Acceleration must be rewritten as:

$$a = \frac{dv}{dt}$$

yielding

$$F = m\frac{dv}{dt}$$

which is the form of $F = ma$ we will use. The expression for the kinetic energy stored in a mass, the well known:

$$\mathbf{E}_m = \frac{1}{2}mv^2$$

is written in terms of our power variable velocity.

The most familiar expression for a spring is:

$$F = Kx$$

but this isn't written in our power variables F and v. Here we must differentiate both sides with respect to time so that we can differentiate displacement x to yield velocity:

$$\frac{dF}{dt} = K\frac{dx}{dt}$$

$$\frac{dF}{dt} = Kv$$

You have most likely not worked with this form of the constitutive equation for a spring before ME 352.

The most common form of the equation for the elastic strain energy stored in a spring is:

$$\mathbf{E}_K = \frac{1}{2}Kx^2$$

but this expression is not in terms of our power variables F and v either. We can write it in terms of F by rearranging:

$$F = Kx$$
$$x = \frac{F}{K}$$

which, when substituted into the energy equation, yields:

$$\mathbf{E}_K = \frac{1}{2}Kx^2$$

$$\mathbf{E}_K = \frac{1}{2}K\left(\frac{F}{K}\right)^2$$

$$\mathbf{E}_K = \frac{F^2}{2K}$$

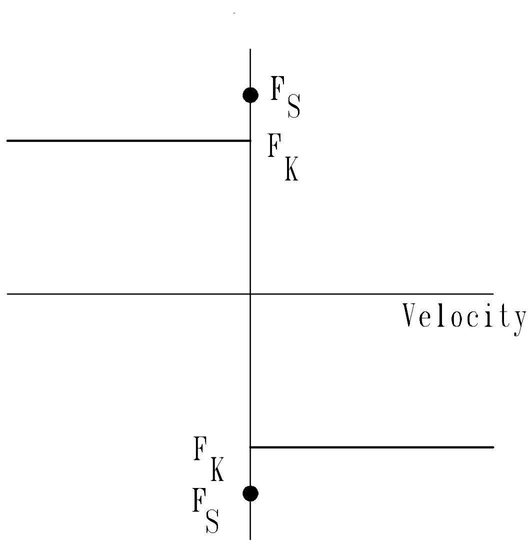
This is also most likely an unfamiliar form.

The mass of an object can be determined with great precision. It can be more difficult to establish a "spring constant" or "spring rate" K for an object because the relationship between force and elastic deformation may be non-linear, either because the stress-strain properties of the material are non-linear (e.g. aluminum) or because the object may deform in a way that alters the stress state in the object. We will work with linear elemental models so that we can solve the differential system equation we derive. Consequently, our spring constants may be linear approximations of non-linear components.

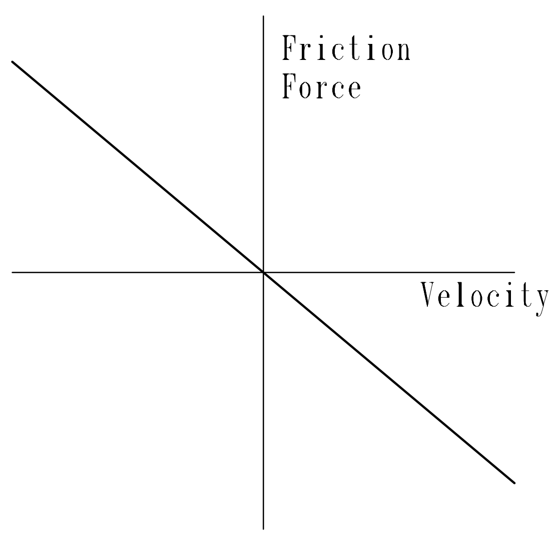
The frictional properties of a system will generally be the greatest uncertainty in our model for a number of reasons. To start with, there are many different types of friction. The most familiar is dry friction between two surfaces. A very useful model consists of static and kinetic friction coefficients, which are constants. Although this model is very useful, it isn't very realistic. Shouldn't the relationship between the normal and tangential force acting on depend on the displacement of the surfaces, not to mention the wear, temperature, lubrication, cleanliness, and chemical alteration of the surfaces? It does; friction is dependent on all these factors.

A more fundamental problem for us is we only have analytical solutions for the system equations which are ordinary linear differential equations with constant coefficients. “Ordinary” means ordinary derivatives, i.e. derivatives of only one variable, as opposed to partial derivatives. “Linear” means that the elemental equations can not have a functional relationship between the power variables, or their derivatives, which is anything but a proportionality. “Constant coefficients” means that all of the terms which make up the coefficients must be constants. Since the coefficients are comprised of elemental parameters; the elemental parameters must be constants. These restrictions prevent us from using realistic friction models, non-linear springs, or any other realistic model of an element in system models, unless we solve the resulting system equation numerically using Mathcad (which we will) or Mathematica, or with dynamic simulation software such as VisSim.

Consider the static and kinetic model of Coulomb friction. The model consists of two horizontal lines for $v \neq 0$ and two points for v = 0. It yields four different values of friction, two for the static case and two for the kinetic case, which depend on the direction of motion, since the frictional force opposes motion. We need to simplify this model to use it analytically because it is non-linear; it consists of two lines (and two points) rather than one straight line. The first thing we will do is neglect the case of static friction (the two points) since we are developing a dynamic model. If a mechanical system isn’t moving then it isn’t dynamic. The response either hasn’t started yet or it has reached steady-state. Further, it is difficult to implement mathematically. How is our model to know which way to apply a static force to oppose motion which hasn’t happened yet? We can model kinetic friction, if we know the direction of motion, by using a force source to apply a constant force to represent the kinetic friction resisting motion. This method works well for first order dynamic systems, since they can not oscillate and we will know the direction of the motion. The only trouble is that we can not turn off the kinetic friction when the velocity reaches zero. We need to keep this in mind interpreting the results of our model.



Static and Kinetic Coulomb Friction Model



Ideal Viscous Friction Model

There is a type of friction that can be easily and accurately represented in a dynamic model. Lubrication provides a thin fluid film between solid surfaces which prevents Coulomb friction because the fluid pressure keeps the surfaces out of contact. Under the proper combination of geometry, lubricant, and relative velocity between the two surfaces, the shear force required to move one surface parallel to the other is proportional to the shear rate, where the shear rate is equal to the difference in velocity of the two surfaces divided by the distance between them. We will call this ideal viscous friction. We will call the proportionality constant b, or B, the "damping" coefficient. The elemental equation which describes ideal viscous friction is:

$$F = bv$$

Note that this is an algebraic equation, not a differential equation. It has the same functional form as the equation for electrical resistance. The force in this expression is friction since the work done to displace the surfaces parallel to each other is dissipated in the fluid as heat and is lost from the system.

Ideal viscous friction has been used to create mechanical components specifically designed to dissipate energy. These devices are called dashpots. A dashpot dissipates kinetic energy in fluid flow through a gap or an orifice. The working fluid may be either oil or air. A familiar example is the use of a dashpot and a spring in parallel in a door closer. The spring provides strain energy to close the door. The dashpot provides a force proportional to the velocity drop across it (i.e. the velocity of the door relative to the door frame) and has the effect of slowing the motion of the door so that it closes without slamming.

Damping need not be provided by a mechanical dashpot. Often polymeric (plastic) or elastomeric (rubbery) materials are used in design to add damping to a component. An example is the use of elastomers in design are the motor mounts in an automobile. An elastomer has both elastic strain energy storage and damping properties.

The elemental and energy equations for translational mechanical systems written in terms of the power variables force and velocity can be summarized as:

	Elemental Equation	**Energy Equation**
Mass	$F = m\frac{dv}{dt}$	$\mathbf{E}_m = \frac{1}{2}mv^2$
Spring	$\frac{dF}{dt} = Kv$	$\mathbf{E}_K = \frac{F^2}{2K}$
Damper	$F = bv$	None. Energy Dissipater

Energetic mechanical systems generally have sources to supply power, although some operate from stored strain energy (e.g. children's toys and the mechanism that deploys the solar panels of a satellite). Sources supply power to a system while allowing control of the time history of one, but not both, of the two power variables. A force source is the most familiar to engineering students, since they have seen a vector labeled F or F(t) acting on free body diagrams. What is not clear from the vectorial representation is that the device which supplies that force has to displace with the displacement of its point of application. If it were to loose contact, then it could not apply the force. Consequently, a force source must also be capable of supplying motion (e.g. velocity). Another aspect unclear in the vectorial representation is that the force source can't float in space and apply force to an object, which is the way a force vector is portrayed. A force source must react against something, since there must be action and reaction. If a force vector is shown floating in space, the reaction is assumed to be provided by ground.

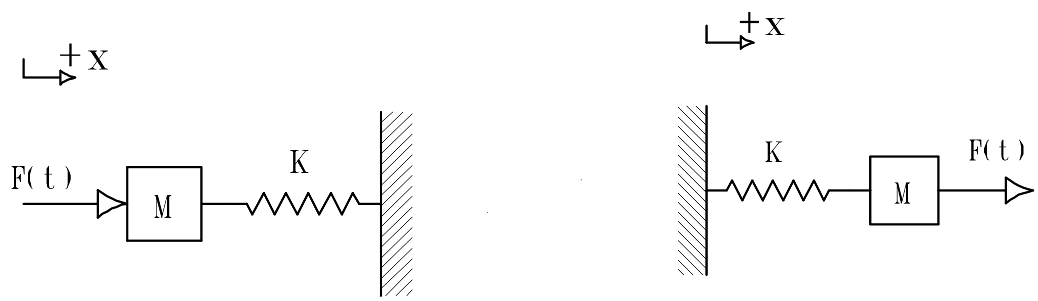
Sources must be able to supply enough power such that they can maintain the specified magnitude of the power variable. The more powerful the device, the more nearly perfectly it can behave as a source. On earth, gravity is a perfect source for a force acting on a mass, since the gravitational force is independent of the velocity of the mass. A powerful hydraulic piston is another good force source. Velocity sources are also commonly used in mechanical design. A velocity source is any mechanical device which has enough power to provide a proscribed motion at a specified velocity. Examples of velocity sources include cam followers and mechanical linkages. They must be powerful enough that they can push their point of application hard enough to follow their specified motion. Some devices can be either a force source or a velocity source, depending on how they are controlled. The Instron testing machine has a 50,000 lb. hydraulic piston which can be placed under either force or velocity control.

Schematics of Translational Mechanical Systems

A schematic of an energetic model of a translational mechanical system uses interconnected symbols to represent springs, masses, dampers, and force and velocity sources. It is important to keep clearly in mind that the schematic represents a "lumped parameter" model of an energetic system. The energetic properties represented by the elements in the schematic are not all of the properties of the system. They are the dominant properties that must be included to model the behavior of the system with the precision needed for the particular engineering analysis being performed. We will not have time in ME 352 to explore dynamic modeling. We will in ME 479.

First, lets introduce the symbols and sign convention we will use starting with a translational spring-mass system. Signs are a problem in system dynamics because so many different, and contradictory, sign conventions are used in mechanical engineering. The sign conventions for force are good examples. There is are two different sign conventions, depending whether the analysis is static or dynamic. A static analysis is used in stress calculations. In this circumstance, a positive force is defined as causing tension in a member, regardless of its orientation in space. In a dynamic analysis, a positive force is defined as causing the acceleration of a mass in the positive direction. In systems dynamics, we need to deal with both situations, the deformation of elastic material caused by force stores strain energy and the velocity of a mass caused by the acceleration due to a force stores kinetic energy. We need to choose one of the two sign conventions, and we will use the convention that a positive force causes acceleration in the positive direction. We will need to exercise care in interpreting the results of our analyses. We must refer to our graphical definition of the positive direction on our mechanical schematic to determine whether a positive force places a spring in tension or compression.

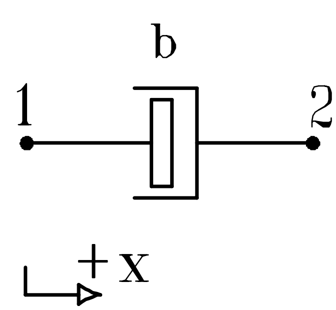
Consider the two similar spring-mass systems below, which differ in the how the spring is connected to ground:



System 1 System 2

In System 1, positive displacement of mass M places the spring in compression. Whereas, in System 2, positive displacement of mass M places the spring in tension. The symbols used above for a mass and a spring are those you have seen previously in Physics. Note that the symbols are also identified with the elemental parameters, M for mass and K for spring constant.

The symbol for damping is a dashpot, which may be unfamiliar. A translational dashpot is a piston-cylinder device. The piston and cylinder can move independently of one another, but must pump fluid into or out of the cylinder to do so. The magnitude of the velocity difference between the piston and the cylinder depends on the axial force acting to extend or compress the dashpot, $F_b = bv_{12}$ for the dashpot below. In this case, if $v_1 > v_2$, then $v_1 - v_2 = v_{12} > 0$ and a force that compresses the dashpot is positive. If $v_1 < v_2$, then $v_1 - v_2 = v_{12} < 0$ and a force that extends the dashpot is negative. As with a spring, this is not a rule but depends on the assumed positive direction for displacement and velocity and on how the element is connected in the system. The lines extending from the piston and cylinder to nodes 1 and 2 are the mechanical equivalent of an ideal conductor in an electric circuit. These lines represent ideally rigid and massless rods. Since they are rigid, they do not deform and cannot store elastic strain energy. Since they are massless, they cannot store kinetic energy. All they can do is transmit force. The velocity of the piston is equal to the velocity of node 1 and the velocity of the cylinder is equal to the velocity of node 2. These may be different velocities, since the piston and cylinder can move independently.



Schematic Symbol for Translational Damping.

Rotational Mechanical Systems

We can use a one-to-one set of analogies to relate rotational mechanical systems to translational mechanical systems because the mechanisms of energy storage and dissipation are physically similar. Power in a rotational system is

$$\mathbf{P} = T\Omega$$

where T is torque and Ω is angular velocity. We will formulate systems equations in SI units. Torque will be in Newton-meters and angular velocity will be in radians / sec. Be careful working with angular velocity because radians are dimensionless since it is angular measure defined as circular arc length divided by circumference. A revolution and a cycle are also dimensionless. If you forget to multiply by 2π radians / cycle or 2π radians / revolution, you will have large numerical errors.

The easiest way to introduce the equations for rotational systems is by direct analogy to translational systems:

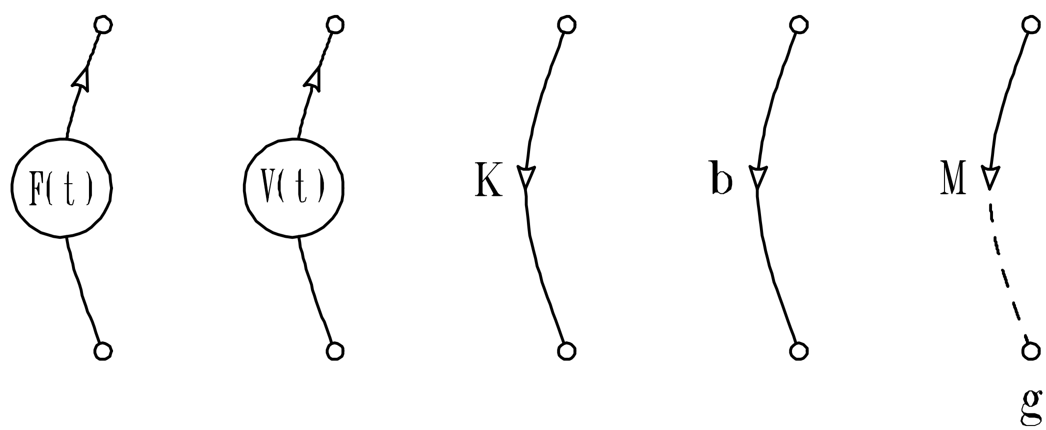
	Elemental Equation	Energy Equation
Kinetic Energy Storage		
Mass	$F = m\dfrac{dv}{dt}$	$\mathbf{E}_m = \dfrac{1}{2}mv^2$
Rotational Inertia	$T = J\dfrac{d\Omega}{dt}$	$\mathbf{E}_J = \dfrac{1}{2}J\Omega^2$
Strain Energy Storage		
Translational Spring	$\dfrac{dF}{dt} = Kv$	$\mathbf{E}_K = \dfrac{F^2}{2K}$
Rotational Spring	$\dfrac{dT}{dt} = K\Omega$	$\mathbf{E}_K = \dfrac{T^2}{2K}$
Energy Dissipation		
Translational Damper	$F = bv$	None. Energy Dissipater
Rotational Damper	$T = b\Omega$	None. Energy Dissipater

Note that the systems are analogous to the extent that the same symbols are used for the spring constants and damping coefficients in the two types of systems. The text uses a subscript R for rotation, to distinguish K and K_R and b and b_R, but this is not necessary. It is clear from the variables used in an equation whether it is rotation or translation.

Drawing Linear Graphs from Mechanical Schematics

An electrical schematic is a graphical network model of an energetic system. We can draw similar network models, called "linear graphs", of any type of energetic system by using the analogies between the through variables and across variable. The first step in drawing a linear graph is to **FIND THE NODES OF DISTINCT VALUES OF THE ACROSS VARIABLE** on the schematic of the system. Once you have identified the nodes on the schematic, then draw the linear graph by first drawing and labeling the nodes and then adding the elements between their respective nodes.

The across variable in mechanical systems is velocity. Springs and dampers have a node a both ends of the element. The only tricky part of drawing a linear graph of a mechanical system is that the velocity of inertias must be referenced to an inertial reference, which we will call ground. All energy storage and energy dissipation elements in a linear graph have two nodes. However, an ideal mass has a single velocity because it is rigid. There is only one velocity node on the mass. The other velocity node associated with the mass is the velocity of the ground node. The linear graph element is drawn between the node representing the velocity of the mass and the ground node. This is confusing at first because it appears to imply that the force applied to the mass acts to ground, as in the cases of a damper or spring attached to ground where the ground must supply a reaction force. The ground does not supply a reaction force for a mass, only an inertial velocity reference. All of the force flowing into a mass element acts to accelerate the mass. Note, all mechanical systems which contain a mass must also have an inertial reference (ground). Sometimes you will see half of the line representing a mass dashed, which is intended to convey that the force is not transmitted to ground.

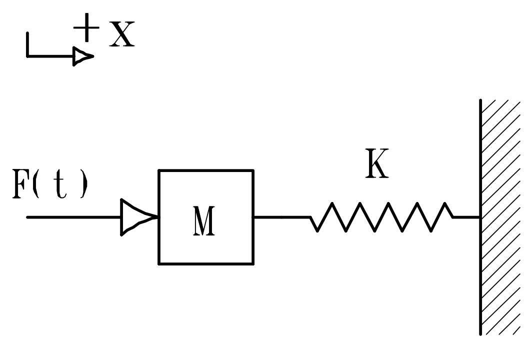


Force Source Velocity Source Spring Damper Mass

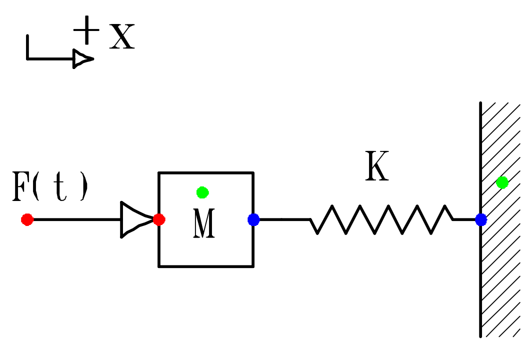
The lines of a linear graph represent the through variable, which is force in a translational mechanical system. The arrows shown on the linear graph elements above indicate the assumed positive direction of the through variable flow in that element. The assumed positive direction of through variable flow is used in the continuity equations, to determine which flows are into and out of a node, and in the elemental equations, to determine the assumed direction of the drop in the across variable velocity. The across variable drop is in the same direction as the assumed positive direction of the through variable, except in sources. Power sources differ from "passive" elements which store or dissipate energy. Sources "pump" energy up hill; the velocity drop is in the opposite direction from the through variable flow. Source symbols differ from the symbols for other element. The convention from electrical circuits is used and sources are identified by a circle containing the power variable under our control.

Example Linear Graphs Drawn from Translational Mechanical Schematics

Example 1: Spring-Mass System.

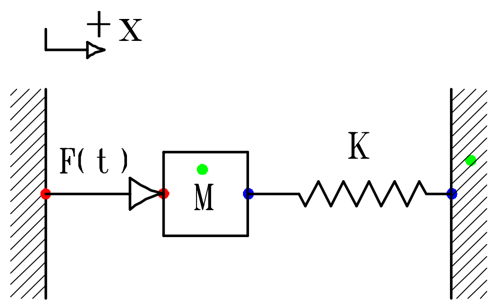


Step 1: Locate all possible nodes of the across variable velocity. Each element has two nodes. All elements except masses have nodes at either end. The nodes at either end of the force source are shown in red and at either end of the spring are shown in blue. Masses have one node on the mass and the other at ground, shown in green.

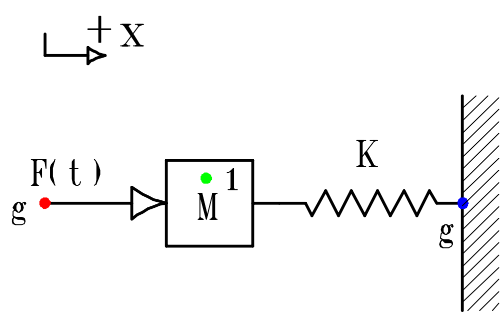


Step 2: Identify which of the possible nodes are redundant and either give them the same name or eliminate them. The mass has the three possible nodes shown, but it can only have one velocity because an ideal mass must be rigid since it can not store elastic strain energy. The two of the three nodes are redundant since the three nodes represent one velocity. How many nodes are shown at ground velocity? The nodes from the spring (blue) and the mass (green) are

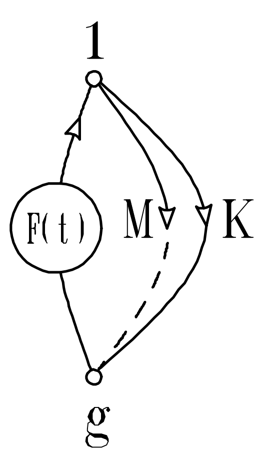
obvious. Less obvious is the reaction of the force source. The left-hand node on the force source is the node that the source reacts against. Since it is not shown to be a dynamic element in the system, it must be ground. An alternative schematic for the system, which shows the reaction of the force source, is:



This system has only two unique velocity nodes, the velocity of the mass and ground.

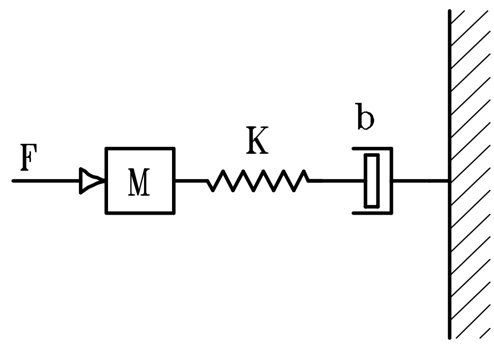


Step 3: Draw the linear graph by drawing and identifying the across variable nodes and then adding the elements between the appropriate nodes. There are just two nodes, Node 1 and ground, so all of the elements act in parallel.

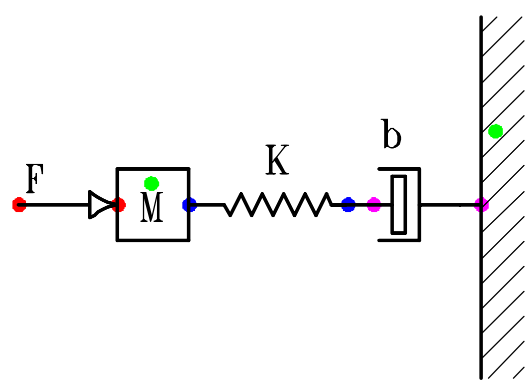


This linear graph says that the force applied by the force source to node 1 divides between the mass and the spring, which agrees with our physical understanding of the system. Assume the system is initially at rest when a step input of force $F(t) = Fu_s(t)$ is applied to the system. Initially, all of the force acts to accelerate the mass. However, when the mass begins to move the spring compresses, and begins to exert force. Eventually, the force is the spring equals the applied force, leaving none to accelerate the mass. This system will actually oscillate forever because there is no damping in it to dissipate energy. This is an unrealistic model because all real systems have damping, but it can be a useful model as it would provide very accurate results for many aspects of lightly damped spring-mass systems.

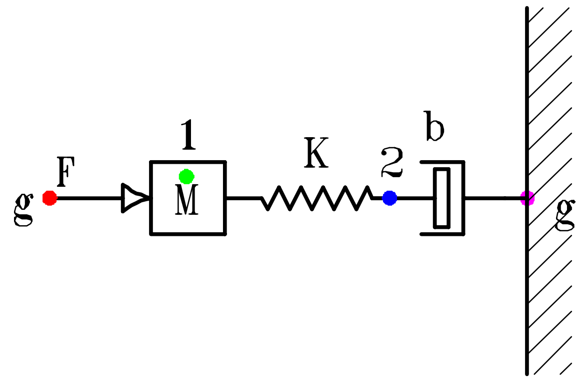
Example 2: Spring-Mass-Damper System.



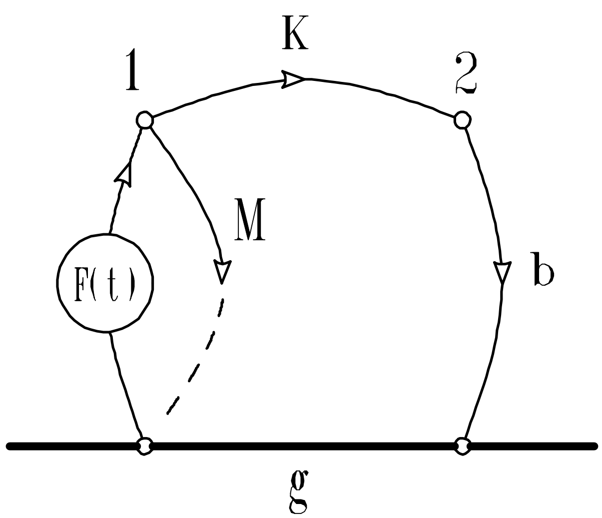
Step 1: Locate all possible nodes of the across variable velocity.



Step 2: Identify which of the possible nodes are redundant and either give them the same name or eliminate them.



Step 3: Draw the linear graph by drawing and identifying the across variable nodes and then adding the elements between the appropriate nodes



Note that there are two ground nodes in this linear graph. The horizontal line at the bottom is the "ground plane". Every node on the ground plane is ground. We could use a single ground node, but two are used to spread out the graph and make it easier to read.

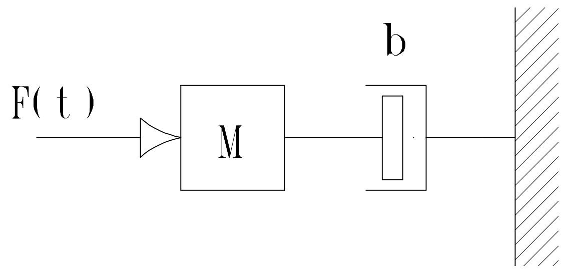
Problem Set No. 3
Due Monday, February 12, 2001

Reading: Rowell and Wormley, Chapters 3 and 4.

Problems: Rowell and Wormley, Problems 3.9 and 4.1.

Problem 1: A translational mechanical system consisting of a force source, a mass, and damper is shown in the schematic below. The system is at rest at time $t < 0$. A step input of force F is applied to the mass at time $t = 0$.

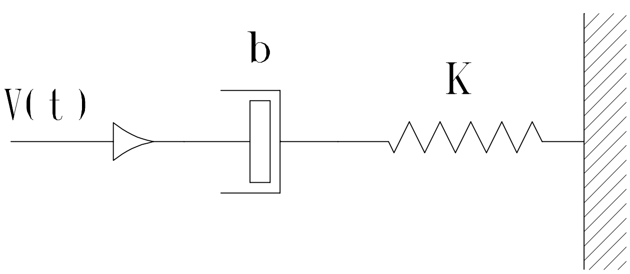
Draw the linear graph that represents this system. Derive , solve, and sketch the velocity of the mass, using $b = 2$, $M = 5$, and $F = 10$.



Problem 2: A translational mechanical system consisting of a velocity source, a spring, and a dashpot (damper) is shown in the schematic below. The system is relaxed at time $t < 0$. A step input of velocity V is applied to the damper at time $t = 0$.

(a) Draw the linear graph that represents this system.

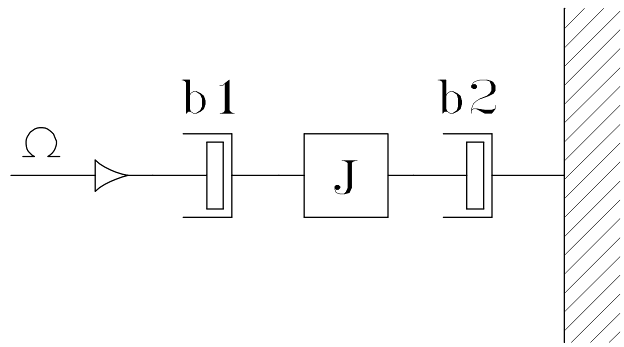
(b) Derive, solve, and sketch the force in the spring and the velocity drop across the spring using $b = 2$, $K = 4$, and $V = 10$.



Problem 3: A rotational mechanical system consisting of an angular velocity source, two dampers, and a rotational inertia is shown in the schematic below. The system is at rest at time $t < 0$. A step input of angular velocity Ω is applied at time $t = 0$. Using $b1 = 1$, $b2 = 2$, $J = 4$, and $\Omega = 10$, derive , solve, and sketch:

a) The angular velocity of the inertia.

b) The torque in damper b_1.



Problem 4: A translational mechanical system consisting of a force source, two dampers, and a spring is shown in the schematic below. (Note: the vertical bar that the applied force acts on has no parameters attributed to it. Consequently, it has no energetic properties. It is the proverbial "massless, rigid bar" can neither store nor dissipate energy. It is an ideal force conductor. This is a translational system, so the bar cannot rotate.)

The system is relaxed at time $t < 0$. A step input of force F is applied at time $t = 0$. Using $b1 = 1$, $b2 = 2$, $K= 3$, and $F = 10$, derive, solve, and sketch the responses of:

a) The force in the spring.

b) The force in damper b_1.