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Lafayette College
ME 352: Dynamics of Physical Systems and Electric Circuits

Problem Set No. 2
Due Friday, February 2, 2001

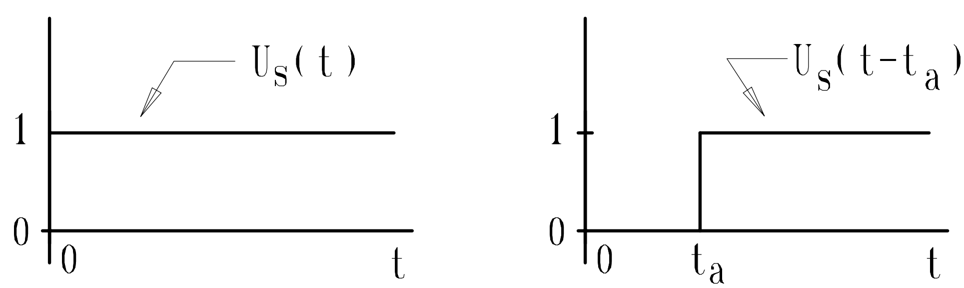
Reading: Rowell & Wormley: Chapter 1 (omit Example 1.3), Chapter 2.
Introductory Mathcad Tutorial on the ME 352 Home Page.

First Order Step Responses: For the time being, we will focus on the response of first order systems to step changes in the input (or source) variable.

The unit step function $u_s(t)$ can have one of two values:

$$u_s(t) = 0 \quad \text{for} \quad t < 0$$
$$u_s(t) = 1 \quad \text{for} \quad t \geq 0$$

Described verbally, the unit step function equals zero before it is turned on. It turns on when the argument equals zero and remains equal to one forever after. The name "step" is used because a time plot of the function resembles a stair step. The function can be turned on at any arbitrary time t_a by subtracting that time from the independent variable t so that the argument of the unit step function remains negative until that time. Graphically, subtracting t_a from t shifts the unit step function forward in time.



Unit Step Functions

In practice, we can not generate ideal (perfect) step inputs because we can not make any energy flow change instantaneously without infinite power. Fortunately, we can model many phenomena as step changes. The criterion is that the input variable changes from zero to a constant value over a time which is small relative to the time scale of the response of the system.

The response of a system to a step input reveals the underlying nature of the dynamic system. Dynamic systems with one independent energy storage element can only have two basic responses a step input. The system can either monotonically gain energy from or lose energy to the source. There can not be oscillations of the system caused by energy flow between energy storage elements within the system since there is only one storage element.

Systems with only one independent energy storage element are called "first order" because the equation relating the input and output variables (the system equation) is a first order differential equation. If the equations which describe the elements of the system are linear equations then the system equation will be a linear differential equation and can be solved by determining the unknown coefficients of a general solution. The step response of all of the power variables in a first order system is either a stable exponential growth or decay. In a given system, some variables will grow while others decay. Importantly, all of the changes occur at the same rate. The step responses of all of the variables in the system have the same time constant.

There are only four general solutions for the step response of a first order system; two growth equations and two decay equations. The output variable can grow to a constant value from either zero or a non-zero initial value. The output variable can either decay from an initial value to zero or to a non-zero constant value.

First, the bad news. As part of your engineering knowledge, you must know how to solve a first order differential equation given a step input. Now the good news. All four of the solutions are based on the same exponential function, $e^{-\left(\frac{t}{\tau}\right)}$.

Lets explore this function. Both time t and the time constant τ are positive. Consequently, the exponent $-\left(\frac{t}{\tau}\right)$ is negative.

Recall that a negative exponent expresses an inverse fractional relationship:

$$e^{-\left(\frac{t}{\tau}\right)} = \frac{1}{e^{\left(\frac{t}{\tau}\right)}}$$

(By the way, remember what the natural log base e equals? If not, use your calculator, $e^1 = 2.72$.)

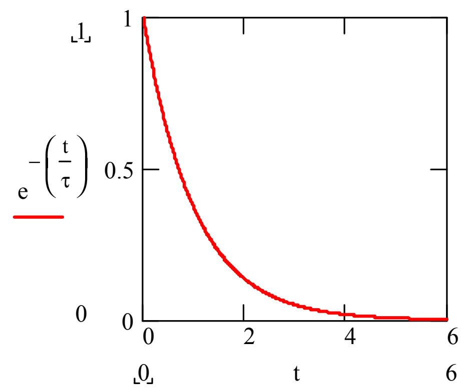
Evaluate $e^{-\left(\frac{t}{\tau}\right)}$ at $t = 0$ and $t = \infty$:

For t = 0: $$e^{-\left(\frac{0}{\tau}\right)} = \frac{1}{e^{\left(\frac{0}{\tau}\right)}} = \frac{1}{e^0} = \frac{1}{1} = 1$$

For $t = \infty$: $$e^{-\left(\frac{\infty}{\tau}\right)} = \frac{1}{e^{\left(\frac{\infty}{\tau}\right)}} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$$

Lets also evaluate it at the times which are the first five multiples of the time constant; t = τ, 2τ, 3τ, 4τ, and 5τ.

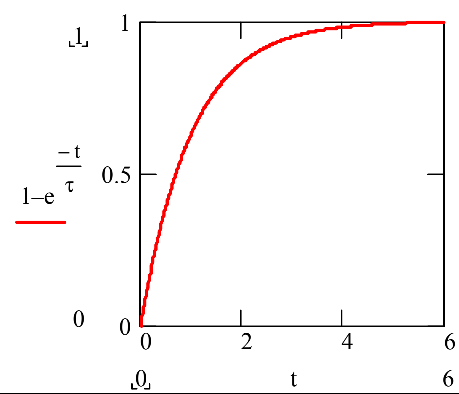
$\frac{t}{\tau}$	$e^{-\left(\frac{t}{\tau}\right)}$
0	1.00
1	0.37
2	0.14
3	0.05
4	0.02
5	0.01
∞	0



The solution for an exponential decay to a non-zero constant value is formed by adding a constant to shift the exponential decay curve vertically, $f(t) = C_1 e^{-\left(\frac{t}{\tau}\right)} + C_2$, where C_1 is the range of the decay and C_2 is the final constant value.

The solution for a stable exponential growth to a constant value has the inverse shape of the exponential decay solution. It is formed by subtracting the exponential decay function from 1, i.e. $\text{growth}(t) = 1 - e^{-\left(\frac{t}{\tau}\right)}$

$\frac{t}{\tau}$	$e^{-\left(\frac{t}{\tau}\right)}$	$1 - e^{-\left(\frac{t}{\tau}\right)}$
0	1.00	0
1	0.37	0.63
2	0.14	0.86
3	0.05	0.95
4	0.02	0.98
5	0.01	0.99
∞	0	1



This exponential function is called a stable exponential growth because it reaches a constant final value. Unstable exponential growth has a positive exponent and grows to infinity.

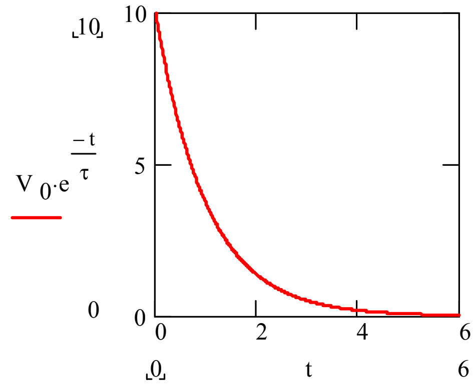
Note that $e^{-1} = \frac{1}{e} = \frac{1}{2.72} = 0.37$. Consequently, the both the growth and decay functions get 63% closer to their final values over each time period equal to one time constant τ. The time constant τ provides the time scale of the response. Large time constants lead to slow responses. Small time constants lead to fast responses. The vertical (or magnitude) scaling is done by multiplying the growth or decay functions by the magnitude of the change from $t = 0$ to $t = \infty$.

Using V(t) for the output variable, $\tau = 1$, and $V_0 = V_{SS} = 10$, the basic forms of the step response are:

Decay to Zero:

$$V(t) = V_0 e^{-\frac{t}{\tau}}$$

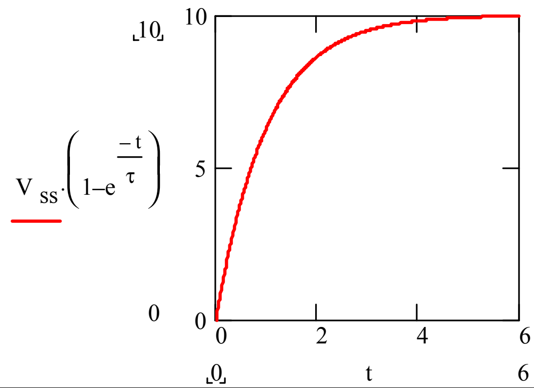
where V_0 is the initial value and τ is the time constant.



Stable Growth from Zero:

$$V(t) = V_{SS}(1 - e^{-\frac{t}{\tau}})$$

where V_{SS} is the final, or steady-state, value.



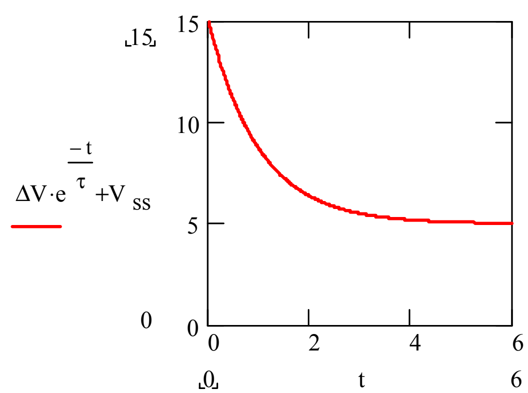
Note: V_0 and V_{SS} can be negative. We have a prejudice in favor of positive numbers, but a variable can decay from or grow to a negative value.

The two variations of the step response, (1) decay to a non-zero steady-state value and (2) stable growth from a non-zero initial value, are created by adding constants to the expressions for decay to zero or growth from zero.

Decay to Non-Zero Steady-State Value:

$$V(t) = \Delta V e^{-\frac{t}{\tau}} + V_{SS}$$

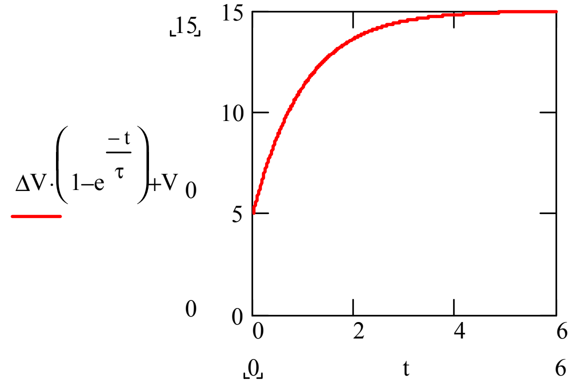
where $\Delta V = V_0 - V_{SS}$



Stable Growth from Non-Zero Value:

$$V(t) = \Delta V(1 - e^{-\frac{t}{\tau}}) + V_0$$

where $\Delta V = V_{SS} - V_0$



Note: V_0 and V_{SS} need not have the same sign.

How does one determine the step response for a specific first order system equation?

Step 1: Derive the system equation that relates the input and output variables. (Remember to always check units.)

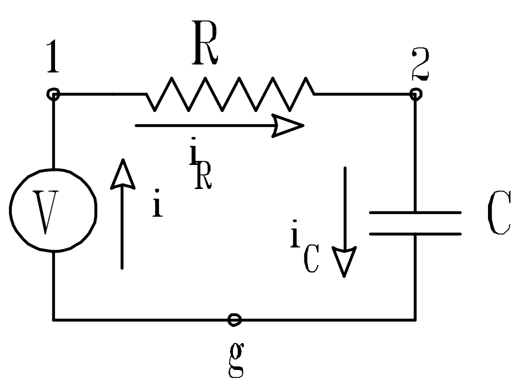
Step 2: Put the system equation into "time constant" standard form.

Step 3: Determine the time constant and the final value of the output variable from the system equation in time constant standard form.

Step 4: Use the energy equation for the energy storage element and the equation list used to derive the system equation to determine the initial value of the output variable at time = 0+, the instant immediately after the input is applied.

Step 5: Compare the initial and final values of the output variable to determine which of the four possible step responses occurs.

EXAMPLE: The RC circuit shown in the schematic below is de-energized before a step input of 10 VDC is applied at time t = 0. Use R = 2 and C = 3. Determine (a) the voltage across the capacitor and (b) the current through the resistor.



Follow the steps of engineering problem solving as detailed in the Big Picture:

Step 1. Draw a picture.

Step 2. Write mathematical statements of relevant physical truths.

Step 3. Reduce the mathematical statements, eliminating all unknown variables except the input and output variables, check the units, and then solve the equation (or system of equations) for the output variable(s).

Step 4. Evaluate the solution. Iterate if necessary.

Step 1. Draw a network representation of the dynamic system. Find and identify the nodes of the across variable. Name and indicate the positive direction of the through variables.

A circuit schematic is a picture. It is a network representation of the circuit. All you need to complete the picture is annotate to it by defining the power variables (voltage and current) for each element. Voltage is the “across” variable measured from node to node across an element. The lines between the elements are assumed to be ideal conductors which neither store nor dissipate energy. There is no voltage drop in an ideal conductor. There are time varying voltage drops across each dynamic element. An easy way to identify the nodes of the across variable in a network is to place a node at either end of every element in the network and then combine all nodes which are on the same ideal conductor, since those nodes all have the same value of the across variable. The names assigned to the nodes are arbitrary. Please use numbers rather than letters because letters can be confused with elemental parameters, e.g. “C” for capacitance. The one common exception is to refer to one voltage in the circuit as “g” for ground.

The through variable in an electric circuit is current. Current flows through an element. The current through each element must be named and an arrow drawn next to the element to indicate the assumed positive direction of the current. The assumed positive direction of the current is also the assumed direction of the voltage drop when you write the elemental equations. Current flows in the direction of a voltage drop, just as fluid flows in the direction of a pressure drop and heat flows in the direction of a temperature drop, except for current flow through a “source”. A voltage source is analogous to a fluid pump. A pump raises the pressure in the direction of the fluid flow. Similarly, a voltage source raises the voltage in the direction of the current flow. One convention for naming currents is to use the elemental parameter as a subscript, e.g. i_R and i_C are the currents through a resistor and capacitor, respectively. There is only one source in this circuit, so the current through the source can be named i without a subscript.

Step 2. Write mathematical statements of relevant physical truths. Write equations for:

(a) Compatibility of Across Variable Drops around Loops.
(b) Continuity (conservation or equilibrium) of Through Variables at Nodes.
(c) Elemental Equations Relating the Across and Through Variables.
(d) Energy Equations for the System and the Energy Storage Elements.

(a) Compatibility of Across Variable Drops around Loops.

A compatibility (loop) equation is a statement that if you measure drops in potential (voltage, pressure, temperature, or velocity) from one point to the next around a loop, the sum of the potential drops around a loop must sum to zero when you end up back where you started. Write as many compatibility equations as there are "holes" (interior loops) in the network. There is only one loop in this circuit, so we can only write one compatibility (loop) equation. The schematic of Problem 2 has two "holes" so write two of the three possible loop equations. Don't write the third equation. It would be redundant.

We will use nodal subscripts on the across variable as a short hand notation for the drop in the across variable. The voltage drop across the resistor is V_1-V_2 = V_{12}. Important: $V_{21} = V_2 - V_1 = -V_{12}$. Be careful with the nodal subscripts. Reversing their order introduces a sign error. Please do not use the alternative notation "V_R" for the voltage drop across the resistor. This is a legitimate and common notation. The drawback is that this notation introduces additional variables in circuits which have elements connected in parallel between the same two nodes.

Pick a node in the loop and write the compatibility equation by summing the drops in the across variable around the loop. Starting at Node 1, the compatibility equation is:

$$V_{12} + V_{2g} + V_{g1} = 0$$

The same equation can be written by equating the sum of the across variable drops from a node to another along the two paths which follow opposite directions around the loop. Starting at Node 1 and summing voltage drops to Node g in both directions yields:

Compatibility: $$V_{1g} = V_{12} + V_{2g}$$

These are the same equation because $V_{1g} = -V_{g1}$. V_{1g} is preferable because the voltage drop across the source is from Node 1 to Node g, in the opposite direction of the assigned positive through variable. Sources are unique elements because they move the through variable from lower potential to higher potential. Again, think of a pump. The fluid pressure is raised in the direction of the flow. In all other elements, the through variable flows in the direction of the across variable drop.

b) Continuity (conservation or equilibrium) of Through Variables at Nodes.

Continuity equations are conservation for electrical, fluid, and thermal systems and equilibrium equations for mechanical systems. In each case, the continuity equation is written for the through variable. We will write continuity equations for the through variable at the nodes, but if we wished we could draw a control volume on part of the network and write the continuity equation for the through variable flow into and out of that control volume. In Thermodynamics, heat flow is positive into a system and work flow is positive out of a system. We will use a different sign convention, call it the Check Book sign convention. The signs are defined by the arrows drawn on the network indicating the assumed through variable flow. Flows into a node are positive. Flows out of a node are negative. The flows must sum to zero for charge (current) to be conserved. We will always write one less continuity equation than we have nodes, so that we do not write redundant equation. We have three nodes, so we will write two continuity equations.

Continuity: Node 1: $i - i_R = 0$
Node 2: $i_R - i_C = 0$

(c) Elemental Equations Relating the Across and Through Variables.

The elemental equations come from your Official ME 352 Crib Sheet. You will not have to memorize them. The equations on the crib sheet do not reference the variables in your system. You need to write the specific equations that refer to your system, using your variables and assumed positive directions.

The generic equation for every resistance in the universe is V = i R. Our resistor has an assumed positive voltage drop from Node 1 to Node 2 in the assumed positive direction of current i_R.

Resistor: $V_{12} = R\, i_R$

Likewise, the generic equation for capacitance is: $i = C\frac{dv}{dt}$. Our capacitor has an assumed direction for the positive voltage drop from Node 2 to Node g in the direction of the assumed positive direction of current i_C.

Capacitor: $i_C = C\frac{dv_{2g}}{dt}$

Is there an elemental equation that describes the source? No, in fact you can formulate the system equation without knowing anything about the source other that what type it is, a voltage source in this case. You do need to know the time history of the source's input variable in order to solve the system equation, but not for formulate it. An ideal source will deliver any specified time history of the input variable. Note that the power delivered by the source is the product of the input variable voltage, whose time history we specify, and the current through the source, which is determined by the response to the system. We can not specify the time history of the power flow into the system. We can only specify the time history of one of the two power variables of the source. The system determines the other.

(d) Energy Equations for the System and the Energy Storage Elements.

Write equations which express what we know about the energy storage in the system. First, there is only one energy storage in this system, the capacitor. Consequently, whatever energy is stored in the system must be stored in the capacitor:

$$\mathbf{E}_{\mathbf{System}} = \mathbf{E}_{\mathbf{Capacitor}}$$

As a shorthand, we will indicate the total energy in the system as $\mathbf{E}$ without a subscript and the energy stored in any particular storage element as $\mathbf{E}_C$, where the subscript is the elemental parameter of the energy storage element. The equation above can be rewritten as:

$$\mathbf{E} = \mathbf{E}_C$$

From the Official ME 352 Crib Sheet, we know the relationship between the amount of energy stored in a capacitor and the voltage across the capacitor, V_{2g} in this circuit.:

$$\mathbf{E}_C = \frac{1}{2} C V_{2g}^2$$

The variable V_{2g} is called an "energy storage variable". Energy storage variables determine the state of the system. If we know the value of the energy storage variables in a system at any point in time we can calculate all of the other power variables in the system at that moment in time. Energy storage variables are also called "state variables" because they determine the "state" of the system.

Finally, we are told that the system is de-energized before the step input is applied at time $t = 0$.

$$\mathbf{E} = 0, \ \ t < 0$$

We now must consider a subtle but important point. Recall that the unit step function $u_s(t)$ can have one of two values:

$$u_s(t) = 0 \quad \text{for} \quad t < 0$$
$$u_s(t) = 1 \quad \text{for} \quad t \geq 0$$

The step function changes abruptly from 0 to 1 at $t = 0$. It will be useful for us to use the following terminology when we refer to $t = 0$:

$t = 0^-$ is the instant before the step input turns on.
$t = 0$ is the instant when the step input turns on.
$t = 0^+$ is the instant after the step input turns on.

Engineers believe in cause and effect. The response of the system to the input must follow the application of the input. Therefore, our solution (the time history of the response of a power variable in the system to the input) must start at $t = 0^+$, the instant after the input step input is applied at $t = 0$.

Does knowing that the system is de-energized at time $t = 0^-$ give us an initial condition for the solution which starts at time $t = 0^+$? Yes. Power is defined as:

$$\mathbf{P} \equiv \frac{d\mathbf{E}}{dt}$$

Our times $t = 0^-$ and $t = 0^+$ are infinitesimally close:

$$dt = 0^+ - 0^- \approx 0$$

If the change in energy stored in the capacitor is finite over an infinitesimally small time period, then:

$$\mathbf{P} = \frac{\Delta\mathbf{E}}{dt} \rightarrow \infty$$

In words, a finite change in the energy over a infinitesimal time period requires infinite power. We don't have infinite power. Therefore, $d\mathbf{E}$ must be infinitesimal over the time interval from $t = 0^-$ to $t = 0^+$. Consequently, for all practical purposes:

$$\mathbf{E}(0^-) = \mathbf{E}(0^+).$$

Let's summarize our equations:

Compatibility:		$V_{1g} = V_{12} + V_{2g}$
Continuity:	Node 1:	$i - i_R = 0$
	Node 2:	$i_R - i_C = 0$
Elemental:	Resistor:	$V_{12} = R\, i_R$
	Capacitor:	$i_C = C\dfrac{dV_{2g}}{dt}$
Energy:		$\mathbf{E} = \mathbf{E}_C$
		$\mathbf{E}_C = \dfrac{1}{2}CV_{2g}^2$
		$\mathbf{E} = 0,\ t < 0$

Please:

1. Draw a line under your equation list to separate the mathematical statements of physical truth from your possibly erroneous reduction of the equation list to a system equation.

2. State the input and output variables before begining the reduction. This may sound like a trivial step, but people often forget what variables they wish to eliminate part way through the reduction.

3. Remember not to use the Energy equations in the reduction of the system equation. The Energy equations are only used to determine the initial conditions to solve the system equation.

Reduction 1: Input V_{1g}, Output V_{2g}

In general, we can start a reduction with any equation other than an Energy equation or a trivial equation (e.g. a = a). There are no trivial equations in our list. The only equation we can not use other than an Energy equation is the Continuity equation for Node 1: $i - i_R = 0$ because the variable i only appears in this equation. We could not eliminate it by substitution unless we were to write the continuity equation for Node g. We would only use the equation $i - i_R = 0$ if i were our output variable. The most attractive equation in our list to start with is the Compatibility equation because it contains both the input and the output variables:

$$V_{1g} = V_{12} + V_{2g}$$

We wish to retain the input V_{1g} and the output V_{2g}. We need to eliminate V_{12} by substitution. The only rule in the reduction is that all of the elemental equations must be used, since they are the only equations that introduce the elemental parameters. Omitting an elemental parameter from the system equation is equivalent to deleting the element from the system. We may as well use the elemental equation of the resistor for the substitution:

Using: $$V_{12} = Ri_R$$

Yields: $$V_{1g} = Ri_R + V_{2g}$$

We need to eliminate i_R. Knowing we need to introduce the elemental parameter for the capacitor, we first use the continuity equation for Node 2: $i_R - i_C = 0$.

Using: $$i_R = i_C$$

Yields: $$V_{1g} = Ri_C + V_{2g}$$

Finally, using: $$i_C = C\frac{dV_{2g}}{dt}$$

Yields: $$V_{1g} = RC\frac{dV_{2g}}{dt} + V_{2g}$$

This is our system equation. It only contains the input variable V_{1g}, the output variable V_{2g} , and elemental parameters.

Reduction 2: Input V_{1g}, Output i_R

We could start with the compatibility equation again because it does contain the input variable and we saw how to introduce the output variable, but just to prove we can begin the reduction with any equation (except the Energy equations and $i - i_R = 0$), let's start with:

$$i_R - i_C = 0$$

Rewrite as: $$i_R = i_C$$

We need to eliminate i_C which only appears also in the elemental equation of the capacitor:

Using: $$i_C = C\frac{dV_{2g}}{dt}$$

$$i_R = C\frac{dV_{2g}}{dt}$$

Now rewrite the Compatibility equation to eliminate V_{2g}:

$$V_{2g} = V_{1g} - V_{12}$$

$$i_R = C\frac{d(V_{1g} - V_{12})}{dt} = C\left(\frac{dV_{1g}}{dt} - \frac{dV_{12}}{dt}\right) = C\frac{dV_{1g}}{dt} - C\frac{dV_{12}}{dt}$$

$$i_R = C\frac{dV_{1g}}{dt} - C\frac{dV_{12}}{dt}$$

Use the elemental equation for the resistor to eliminate V_{12}:

Using: $$V_{12} = Ri_R$$

$$i_R = C\frac{dV_{1g}}{dt} - C\frac{dRi_R}{dt} = C\frac{dV_{1g}}{dt} - RC\frac{di_R}{dt}$$

$$i_R + RC\frac{di_R}{dt} = C\frac{dV_{1g}}{dt}$$

This is our system equation. It only contains the input variable V_{1g}, the output variable i_R, and elemental parameters. I will rewrite the system equation to have the input variable on the left and output variable on the right:

$$C\frac{dV_{1g}}{dt} = RC\frac{di_R}{dt} + i_R$$

Check Units:

We will now check the units of the system equations. It is counterproductive to check the units of the system equation by using fundamental units. It can be done this way, but our unit check is intended to reveal errors in our reduction. Expressing hybrid systems, such as electromechanical systems (e.g. electric motors) in fundamental units is complicated enough that errors may be introduced. It is far easier to check the units of the system equation by expressing the elemental parameters in terms of the power variables and time. If all terms in the equation have the same units, then we have eliminated many of the possible errors in the reduction.

Start with the elemental equations, dropping the subscripts for convenience. Rearrange them to express the elemental parameter as a function of the power variables and time. The square brackets [ ] are read “units of”. Important: the derivative operator has no units, i.e. $\left[\frac{dV}{dt}\right] = \left[\frac{V}{t}\right]$.

Resistor: V= iR $\qquad [R] = \left[\frac{V}{i}\right]$

Capacitor: $i_C = C\frac{dV_{2g}}{dt} \qquad [C] = \left[\frac{it}{V}\right]$

Unit Check of System Equation 1: $V_{1g} = RC\frac{dV_{2g}}{dt} + V_{2g}$

$$[V] = \left[RC\frac{dV}{dt}\right] + [V]$$

$$[V] = [R][C]\left[\frac{V}{t}\right] + [V]$$

$$[V] = \left[\frac{V}{i}\right]\left[\frac{i\,t}{V}\right]\left[\frac{V}{t}\right] + [V]$$

$$[V] = [V] + [V] \qquad \text{Units check.}$$

Unit Check of System Equation 2: $$C\frac{dV_{1g}}{dt} = RC\frac{di_R}{dt} + i_R$$

$$\left[C\frac{dV}{dt}\right] = \left[RC\frac{di}{dt}\right] + [i]$$

$$[C]\left[\frac{V}{t}\right] = [R][C]\left[\frac{i}{t}\right] + [i]$$

$$\left[\frac{it}{V}\right]\left[\frac{V}{t}\right] = \left[\frac{V}{i}\right]\left[\frac{it}{V}\right]\left[\frac{i}{t}\right] + [i]$$

$$[i] = [i] + [i] \qquad \text{Units check.}$$

The units in each term of both of the system equations are consistent. Note, this does not mean the equations are correct. Unit checks can not reveal errors in dimensionless terms. For example, should a term be $\frac{R_1}{R_2}$ or $\frac{R_2}{R_1}$ or $\left(1+\frac{R_1}{R_2}\right)$? All of these expressions are dimensionless.

Solve the System Equation for the Output Variable:

Step A: Put the System Equation into Time Constant Standard Form to Determine the Time Constant and the Steady-State Value :

The Time Constant Standard Form of a first order differential equation is very useful. It provides two of the three values we need to determine the step response: the time constant τ and the final value of the output variable. Take another look at the form and units of the our first order system equations. Note that each term in the equations must have the same units, or the right side can not be summed and equated to the left side. The term multiplying the first derivative of the output variable must have units of time to cancel the unit of time in the denominator of the derivative. The name “Time Constant” comes from the facts the multiplying term is comprised of constants and is, therefore, a constant, and that it has units of time. The time constant is given the symbol τ.

$$V_{1g} = RC\frac{dV_{2g}}{dt} + V_{2g} \qquad [V] = [R][C]\left[\frac{V}{t}\right] + [V]$$

$$C\frac{dV_{1g}}{dt} = RC\frac{di_R}{dt} + i_R \qquad [C]\left[\frac{V}{t}\right] = [R][C]\left[\frac{i}{t}\right] + [i]$$

Both of our system equations are already in "Time Constant Standard Form" since there is no parameter term multiplying either output variable V_{2g} and i_R. If there were a multiplying term on the output variable, such as in the equation:

$$\alpha V_{1g} = \beta\frac{dV_{2g}}{dt} + \gamma V_{2g}$$

Then divide both sides by γ to clear the output variable of the multiplying term γ, giving Time Constant Standard Form:

$$\frac{\alpha}{\gamma}V_{1g} = \frac{\beta}{\gamma}\frac{dV_{2g}}{dt} + V_{2g}$$

The final or steady-state value of the output variable can also be determined from the system equation in Time Constant Standard Form. "Steady-state" is the state the system is in after the transient effects of the input on the system decay to zero. A system given a step input will eventually reach a constant steady-state output. The magnitude of the constant output can be determined from the system equation in Time Constant Standard Form by setting all of the time derivatives to zero, since the steady-state is constant and the time derivative of a constant is zero.

The steady-state value the output V_{2g} of System Equation 1:

$$V_{1g} = RC\frac{dV_{2g}}{dt} + V_{2g}$$
$$V_{1g} = RC\xcancel{\frac{dV_{2g}}{dt}} + V_{2g}$$
$$V_{1g} = V_{2g}(SS)$$

where V_{1g} is the magnitude of the step input.

The steady-state value of i_R of System Equation 2:

$$C\frac{dV_{1g}}{dt} = RC\frac{di_R}{dt} + i_R$$
$$C\xcancel{\frac{dV_{1g}}{dt}} = RC\xcancel{\frac{di_R}{dt}} + i_R$$
$$0 = i_R(SS)$$

Step B: Use the Equation List to Find the Initial Value of the Output Variable. Always Start With the Energy Equation to Find the Initial Value of the Energy Storage Variable.

Compatibility: $V_{1g} = V_{12} + V_{2g}$

Continuity:
Node 1: $i - i_R = 0$
Node 2: $i_R - i_C = 0$

Elemental:
Resistor: $V_{12} = R\, i_R$
Capacitor: $i_C = C\dfrac{dV_{2g}}{dt}$

Energy:
$$\mathbf{E} = \mathbf{E}_C$$
$$\mathbf{E}_C = \frac{1}{2}CV_{2g}^2$$
$$\mathbf{E} = 0,\ t < 0$$

The initial value the output V_{2g} of System Equation 1:

This is easy because the output variable is the energy storage variable. Using the Energy equations and that finite Power requires $\mathbf{E}(0-) = \mathbf{E}(0+)$:

$$\mathbf{E}(0^-) = \mathbf{E}(0^+) = \mathbf{E}_C(0^+) = \frac{1}{2}CV_{2g}^2(0^+) = 0$$
$$\therefore V_{2g}(0^+) = 0$$

The initial value the output i_R of System Equation 2:

Although i_R is not the energy storage variable, always start the analysis of an initial value by determining the initial value of the energy storage variable. From the above:

$$V_{2g}(0^+) = 0$$

From the equation list:

$$V_{1g}(0^+) = V_{12}(0^+) + \cancel{V_{2g}(0^+)}$$

Use the Elemental Equation for the resistor:

$$V_{1g}(0^+) = R\, i_R(0^+)$$
$$\therefore i_R(0^+) = \frac{V_{1g}(0^+)}{R}$$

Step C: Compare the Initial and Final Values of the Output Variable to Determine the Step Response.

The time constant τ for the step responses of both output variables is the same, $\tau = RC = (2)(3) = 6$. This is not a coincidence. There is only one time constant for a given system. All of the variables in that system change at the same rate.

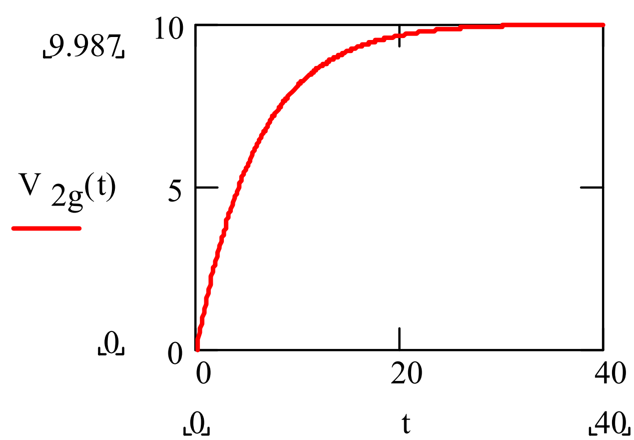
There are only four possible step responses, as detailed above. Compare the initial and final values of the output variables to determine the appropriate response function.

Step Response $V_{2g}(t)$:

$V_{2g}(0^+) = 0$ and $V_{2g}(SS) = V_{1g} = 10$. This response is a stable exponential growth from zero.

$$V_{2g}(t) = V_{2g}(SS)\left(1 - e^{-\left(\frac{t}{\tau}\right)}\right)$$

$$V_{2g}(t) = 10\left(1 - e^{-\left(\frac{t}{6}\right)}\right)$$

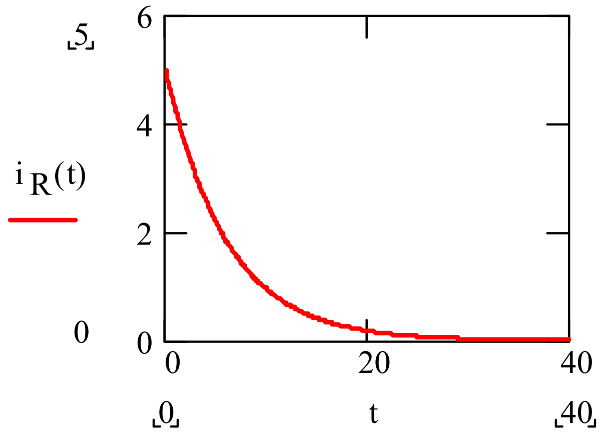


Step Response $i_R(t)$:

$i_R(0^+) = \frac{V_{1g}(0^+)}{R} = \frac{10}{2} = 5$ and $i_R(SS) = 0$. This response is an exponential decay to zero.

$$i_R(t) = i_R(0^+)\, e^{-\left(\frac{t}{\tau}\right)}$$

$$i_R(t) = 5e^{-\left(\frac{t}{6}\right)}$$



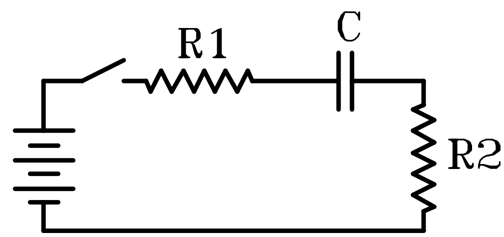
Problem Set No. 2

Rowell and Wormley: Problems 2.5, 2.7, and 2.8.

Problem 1. An electric circuit is shown schematically below. The switch is open and the circuit is de-energized at time t = 0-. The switch is closed at time t = 0 applying a step input voltage V to the circuit. Derive the system equation and check its units for:

a) The voltage across the capacitor.

b) The current through resistor R1.



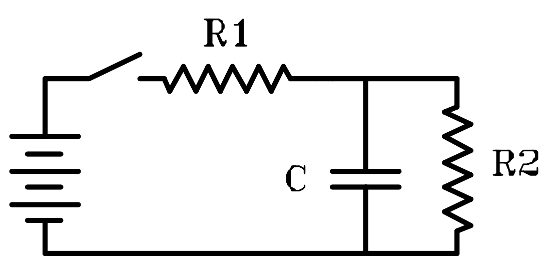
- Solve the system equations using R1 = 1, R2 = 2, C = 3, and V = 10.
- Sketch these responses, or, if you wish, plot the responses using Mathcad.
- How long does it take the voltage across the capacitor to reach 63% of its final value?

Problem 2: An electrical system consisting of two resistors and a capacitor is shown in the schematic below. The system is de-energized at time $t = 0^-$. A step input of voltage V is applied at time t = 0. Derive, check the units, and solve the system equations using R1 = 1, R2 = 2, C = 3, and V = 10 for:

a) The voltage across the capacitor.

b) The current through resistor R1.

c) The current through resistor R2.



Sketch these responses, or, if you wish, plot the responses using Mathcad.