© Dr. Seeler **Department of Mechanical Engineering** **Spring 2001**
Lafayette College

ME 352: Dynamics of Physical Systems and Electric Circuits

THE BIG PICTURE

At this point in your education, it is worthwhile to look for methods and analogies common to your various engineering courses. While each of your engineering courses presents new material, the method of engineering problem solving is essentially the same regardless of the application. This document summarizes the method of engineering problem solving and applies it to a mechanical design problem and a system dynamics problem.

Engineering Problem Solving

Step 1. Draw a picture(s). Annotate it (them) to define all of the variables and parameters .

Step 2. Write mathematical statements of relevant physical truths. Do not use a variable or parameter you have not defined graphically.

Step 3. Reduce the mathematical statements, eliminating all unknown variables except the input and output variables, and solve the equation (or system of equations) for the output variable(s).

Step 4. Evaluate the solution. Iterate if necessary.

Step 1. Draw a Picture.

Graphical representation is the first step in engineering analysis. If you look back at your courses to date, every engineering analysis started with a picture. In Statics, your instructors stressed the need to draw pictures (i.e. free body diagrams) by penalizing you points even if you reached the correct solution. Why such a hard line? Because we use graphics to establish our model and define our variables. Modeling is the most important step in engineering problem solving because even a very approximate solution to a reasonable model is helpful, whereas a very precise solution to the wrong model is dangerous.

In ME 352, we will use "lumped parameter" models. You have, in fact, already drawn many lumped parameter models. Free body diagrams are examples of lumped parameter models since physical properties that are actually distributed, such as mass, are represented by a single parameter. Electrical schematics are also lumped parameter models.

Modeling is the most interesting step in engineering analysis because it requires judgment. For example, electrical resistance is present in all non-super conducting materials. However, in electrical schematics resistance is only indicated for discrete components named resistors in order keep the lumped parameter model has to simple enough to reduce it mathematically. We only include "significant" amounts of resistance. How does one judge significance? Sometimes it is obvious because there is an orders of magnitude difference between the amount of a given physical property in two components of a system. Other times, it may not be at all obvious and may require iterating the model.

Once you have established a model, the reduction and solution is algorithmic, and in engineering practice is done by a computer. Consequently, modeling is the key step in the analysis. If your model (whether it is a free body diagram, electrical schematic, control volume, or lumped parameter model) is wrong, your entire analysis is at best useless, regardless of the sophistication of the reduction and solution. If your model is correct, you can often make the necessary engineering decisions simply by looking at the picture. In more complex situations, you must proceed from a graphical model to a mathematical model.

The graphical model (picture) must be annotated with all relevant parameters and variables before proceeding to the second step of writing equations. If in Step 2, you find you have forgotten a parameter or a variable, do not neglect to add it to the picture. Failure to do so results in errors, particularly sign errors.

Step 2. Write mathematical statements of relevant physical truths.

Many engineering students mistakenly believe that engineers (or engineering students) faced with an unfamiliar problem can think as clearly and efficiently as analyses presented in textbooks. If you have ever asked yourself, "How did the author know to introduce that equation at this point in the analysis?", the answer is that maybe the author didn't, the first time he or she solved the problem. Engineering texts are edited and re-edited to be clear and efficient as possible. Engineers do not solve unfamiliar problems with the efficiency of a textbook analysis.

The general method of engineering problem solving is not to introduce equations which represent physical truths as needed during the derivation of an input-output equation, as is done in textbooks, but to separate Step 2, the process of writing equations that represent the various physical truths of the system, from Step 3, the reduction of those equations to the input-output relationship. Separating these processes into two distinct steps helps both identify all of the relevant physical truths and ease the mathematical reduction, as will be demonstrated in the examples that follow.

What we is need a complete set of independent equations. In most cases, the statements of physical truth in an engineering problem can be grouped into four independent categories, the order of which is irrelevant:

Independent Physical Truths:

1. **Compatibility**

2. **Continuity (Conservation) or Equilibrium**

3. **Material or Elemental Model**

4. **Energy and Power**

1. Compatibility: A fundamental physical truth is that if you go around a loop, you get back to where you started. If you measure your progress around a loop in steps, the steps must be compatible in the sense that the sum of the steps must be equal to the whole. Fluid systems must have pressure drops across components that add to zero when you go around a loop. Electrical systems must have voltage (potential) drops across components that add to zero around loops. Thermal systems must have temperature drops that add to zero around loops. Mechanical systems must have geometric compatibility of displacement, velocity, and acceleration. The deformations of the individual parts must be compatible in that they must add up to the deformation of the whole object.

2. Continuity (Conservation) and Equilibrium: Another category of fundamental physical truth is that matter and energy can either stay put or go somewhere, but they can not just disappear. They are conserved. The conservation principle can be applied by measuring flow rates. In the special case of an element that can not store matter or energy, what flows in must flow out. In fluid systems, if an incompressible fluid flows in one end of a rigid pipe, it must flow out the other. In electrical systems, if current flows in one end of a wire, it must flow out the other. In thermal systems, if heat flows in one end of a metal bar, it must flow out the other.

Equilibrium in mechanical systems is analogous to continuity, if the opposite of the net force which accelerates mass (called the inertial force) is included in equilibrium equations. The inertial force is the force that pushes back at you when you accelerate an object. If the inertial force is included, all force summations equal zero.

3. Material or Elemental Model: Compatibility and Continuity equations are summations which equal zero. There is only one type of variable in each equation. Force summations have only forces. Deformation summations have only deformations. The material or elemental model equations relate two different variables by means of a parameter. The equation which describes a spring element is familiar: $F = k x$, where F is the force in the spring, k is the spring constant (also called the spring rate), and x is the deformation of the spring. This equation relates two different types of variables, force and displacement. The spring constant is the parameter. The equation for a resistor is another such elemental equation: $V = i R$ relates two different type of variables, voltage and current. Resistance is the parameter.

The product of voltage times current is electric power. Hence, voltage and current are called the "power variables" for an electrical system. The elemental equations we will use in ME 352 will be expressed in terms of power variables. Other variables could be used, charge for instance, but they are less convenient because they lead to integral forms of the elemental equations which are more difficult to manipulate than the differential form. We will use slightly unfamiliar forms of the elemental equations F = k x and F = m a in order to express them in terms of the power variables of mechanical systems. Mechanical work, which is the most fundamental definition of energy, is the product of force times displacement in the direction of the force, $\mathbf{E} = F\,x$. Power is defined as the rate at which work is done, or, alternatively, the rate at which energy flows into or out of an energy storage element.

$$\mathbf{P} = \frac{d\mathbf{E}}{dt} = \frac{d(Fx)}{dt} = F\frac{dx}{dt} + x\frac{dF}{dt}$$

The term $x\frac{dF}{dt}$ does not contribute power because the displacement is constant. There must be motion to change the amount of energy in a mechanical system, as we will discuss in lecture. That leaves:

$$\mathbf{P} = \frac{d\mathbf{E}}{dt} = F\frac{dx}{dt} = Fv$$

where the time rate of change of displacement $v \equiv \frac{dx}{dt}$ is velocity. In ME 352, we will view masses and springs as mechanical energy storage elements. The elemental equation for a mass is:

$$F = m\,a$$

which can be written in terms of velocity as:

$$F = m\frac{dV}{dt}$$

Hooke's law:

$$F = kx$$

can be written in terms of velocity as:

$$\frac{dF}{dt} = kv$$

which can be written as:

$$v = \frac{1}{k}\frac{dF}{dt}$$

where the inverse of the spring constant, $\frac{1}{k}$, is called compliance. The uninitiated would not recognize this form as the elemental equation for a spring.

Expressing the equations for mechanical energy storage elements in terms of the mechanical power variables F and v allows us to develop analogies between different types of physical systems because the equations have the same form. Compare the elemental equations for a capacitor and an inductor with those of a mass and a spring:

Mass: $F = m\frac{dV}{dt}$ Capacitor: $i = C\frac{dV}{dt}$

Spring: $v = \frac{1}{k}\frac{dF}{dt}$ Inductor: $v = L\frac{di}{dt}$

We will develop analogies between different types of physical systems which will allow us to create a single model to represent devices which contain different subsystems, such as an electric motor which has both electrical and mechanical elements.

4. Energy and Power: Lastly, we can write statements which express the amount of energy stored in an element or the rate at which power is dissipated by an element. Energy storage equations always contain an elemental parameter and all but thermal capacitance and gravitational potential energy have a squared term. For example, the amount of kinetic energy stored in a mass is:

$$\mathbf{E}_m = \frac{1}{2}mv^2$$

The amount of energy stored in a spring can be expressed in terms of either the applied force or the resulting deformation:

$$\mathbf{E}_k = \frac{1}{2}kx^2 = \frac{F^2}{2k}$$

The amount of electrical energy stored in a capacitor is:

$$\mathbf{E}_C = \frac{1}{2}CV^2$$

But the amount of thermal energy stored in a mass is:

$$\mathbf{E}_T = C_p m T$$

where the product of the specific heat and mass can be expressed as a single parameter C_T, called the thermal capacitance.

Step 3. Reduce the mathematical statements, eliminating all unknown variables except the input and output variables, and solve the equation (or system of equations) for the output variable(s).

By separating Steps 2 and 3, the derivation of the system equation, the differential equation that represents the dynamics of a system, becomes a straight forward, although sometimes laborious, process of reducing the mathematical statements of physical truth to a single equation or to a system of equations. The reduction is an exercise in making substitutions to eliminate unwanted variables, possibly differentiating with respect to time but never integrating. It is literally possible to start the reduction by randomly choosing any compatibility, continuity, or elemental equation, with the exception of "trivial" equations of the form x = x. The energy equations are not used to derive the system equation. They are used to provide the initial conditions to solve the differential equation, as we shall see.

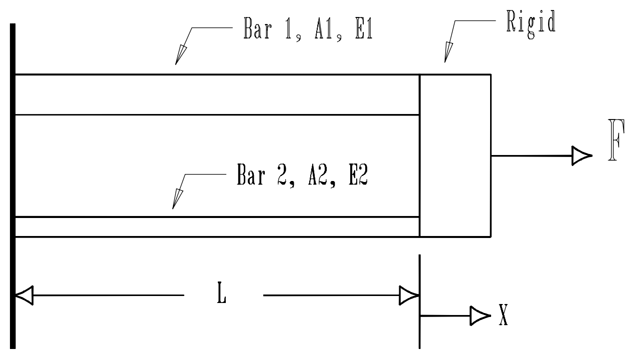
Note the use of the verb "solve", not the verb "integrate". Our purpose is to derive a mathematical model that allows us to predict how a physical system will change with time. Our purpose is not to investigate various techniques of integration. We will be able to represent high order dynamic systems as the superposition of first and second order responses. Consequently, we will only need to solve first and second order ordinary differential equations with constant coefficients. The verb "solve" means to find the solution. Because we are dealing with physical systems, we will already know the forms of the possible solutions. In the case of first order differential equations, there are only two possible responses and "solving" will be simply identifying which response is reasonable and calculating the appropriate coefficients to use in the standard form. Second order dynamic systems have a much broader range of possible responses. Their solution is consequently more involved and will require the use of complex variables. However, we will again know the standard form with which we are working and the solution will involve finding undetermined coefficients.

It is essential to explore the analytical solution of second order systems to understand the response of dynamic systems. However, once that understanding is gained it is much more productive to use numerical solutions. We will use the software Mathcad to solve and plot our system equations. Mathcad will also allow us to model higher order dynamic systems as a set of simultaneous differential equations.

Engineering Problem Solving Examples

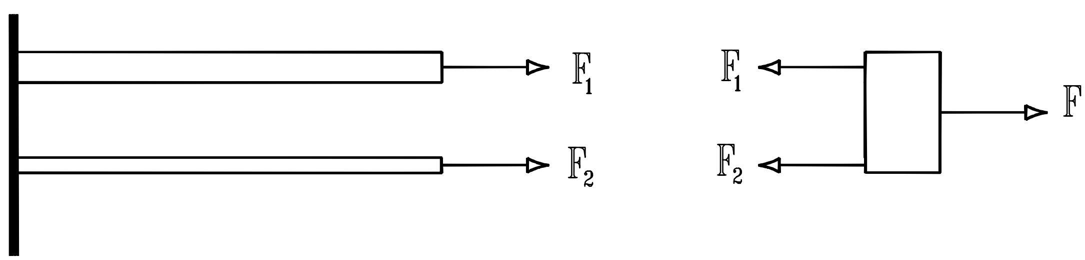
Example 1: Mechanical Design Problem

Two dissimilar metals bars with Young's moduli E_1 and E_2 have the same length L but different cross-sectional areas A_1 and A_2, as shown in the figure below. A force F is applied by an ideally rigid member such that the bars elongate without rotation. Find the stress in bar 1.



Step 1. Draw and annotate a picture (or pictures).

The geometry of the structure is shown in the above figure, but we do need to draw free body diagrams:



Step 2. Write mathematical statements of physical truth.

1. Compatibility: Geometric compatibility tells us that if the bars elongate without rotation then the displacements x_1 and x_2 are equal.

$$x_1 = x_2 = x$$

2**. Continuity or Equilibrium:** The free body diagram allows us to write an equilibrium equation:

$$F = F_1 + F_2$$

3**. Material Models or Elemental Equations:** We can relate stress and strain in both materials.

$$\sigma_1 = E_1\varepsilon_1 \quad \text{and} \quad \sigma_2 = E_2\varepsilon_2$$

We can also write equations that relate displacements to strains and forces to stresses:

$$\varepsilon_1 = \frac{x_1}{L} \quad \text{and} \quad \varepsilon_2 = \frac{x_2}{L}$$

$$\sigma_1 = \frac{F_1}{A_1} \quad \text{and} \quad \sigma_2 = \frac{F_2}{A_2}$$

and write equations for the spring constants (or spring rates) of the bars:

$$k_1 = \frac{F_1}{x_1} \quad \text{and} \quad k_2 = \frac{F_2}{x_2}$$

4. Energy and Power: The amount of strain energy stored each bar is:

$$\mathbf{E}_1 = \frac{1}{2}k_1 x_1^2 \quad \text{and} \quad \mathbf{E}_2 = \frac{1}{2}k_2 x_2^2$$

The total amount of energy stored in the system is the sum of the strain energy in both bars:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

Because the system is elastic, the total energy in the system can be written in terms of the applied force F and the displacement x:

$$\mathbf{E} = \frac{1}{2}Fx$$

Note that the elastic energy $\mathbf{E} \neq Fx$ because F is not constant. F increases linearly with displacement.

3. Reduction and Solution:

Restatement of problem (a): Given the input variable F and the parameters L, A_1, A_2, E_1, and E_2, find the output variable σ_1. We can start with any compatibility, equilibrium, or material model equation. A reasonable choice would be to pick an equation that has either our input or our output variable in it. Let's start with:

$$\sigma_1 = \frac{F_1}{A_1}$$

The only variables we want in our final expression are F and σ_1. F_1 is an unwanted variable. Substitute to eliminate it. Look in your list of equations for one with F_1 in it and use it. It doesn't matter which equation with F_1 you use, except that you don't want to use the same equation twice because that causes you to go in circles. Let's use:

$$F = F_1 + F_2 \quad \text{as} \quad F_1 = F - F_2$$

and substitute:

$$\sigma_1 = \frac{F - F_2}{A_1}$$

F_2 is unwanted. Find an equation in your list with F_2 in it and use it to eliminate F_2. Let's use:

$$\sigma_2 = \frac{F_2}{A_2} \quad \text{as} \quad F_2 = \sigma_2 A_2$$

and substitute:

$$\sigma_1 = \frac{F - \sigma_2 A_2}{A_1}$$

Eliminate σ_2. If we use $\sigma_2 = \frac{F_2}{A_2}$ again, then we will go backwards. Find another expression with σ_2 in it. Use:

$$\sigma_2 = E_2 \varepsilon_2$$

To yield:

$$\sigma_1 = \frac{F - E_2 \varepsilon_2 A_2}{A_1}$$

Eliminate ε_2. Use:

$$\varepsilon_2 = \frac{x_2}{L}$$

To yield:

$$\sigma_1 = \frac{F - E_2 \frac{x_2}{L} A_2}{A_1}$$

Eliminate x_2. Use:

$$x_1 = x_2$$

To yield:

$$\sigma_1 = \frac{F - E_2 \frac{x_1}{L} A_2}{A_1}$$

Eliminate x_1. Use:

$$\varepsilon_1 = \frac{x_1}{L} \quad \text{as} \quad x_1 = \varepsilon_1 L$$

To yield:

$$\sigma_1 = \frac{F - E_2 \varepsilon_1 A_2}{A_1}$$

Eliminate ε1: Use:

$$\sigma_1 = E_1 \varepsilon_1 \quad \text{as} \quad \varepsilon_1 = \frac{\sigma_1}{E_1}$$

To yield:

$$\sigma_1 = \frac{F}{A_1} - \frac{E_2}{A_1 E_1} \sigma_1 A_2$$

We now have an expression that has only the input and output variables and parameters. We are done substituting. We are not yet in what would be considered standard form so we rearrange this equation to isolate the output variable on one side.

$$\sigma_1 = \frac{F}{A_1 + A_2 \frac{E_2}{E_1}} = F\left(\frac{E_1}{A_1 E_1 + A_2 E_2} \right)$$

Now, check the equation. **ALWAYS CHECK THE UNITS!!** If the units of any term in the equation are wrong, then the equation is wrong. These units check, since the units of stress is force/area. Now check for reasonableness, since the equation can be wrong even if the units are correct. This equation tells us that the stress in bar 1 increases if either the area of the bar decreases or the modulus of the bar increases. Is this reasonable? Yes.

Was this the most efficient solution? Probably not, but that is irrelevant. The point is that once you have written a complete set of independent mathematical statements for all relevant physical truths for a system then you can solve for **ANY** output variable. To prove this, let's calculate the equivalent spring rate for the structure. First, what does "equivalent" mean? The equivalent spring rate is the spring rate of a single spring which would provide the same response as the existing structure. There are two complementary methods to calculate the equivalent spring rate because the spring rate appears in both the elemental equation and the energy equation of the spring:

$$F = kx \quad \text{and} \quad \mathbf{E} = \frac{1}{2}kx^2$$

where F is the total force applied to the structure and **E** is the total energy stored in the structure. Let's first start with the energy equation.

$$\mathbf{E} = \frac{1}{2}kx^2$$

Here we have two unwanted variables, **E** and x. Start substituting using equations from the list of physical truths.

$$\mathbf{E}_1 + \mathbf{E}_2 = \frac{1}{2}kx^2$$

$$\frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 = \frac{1}{2}kx^2$$

$$\frac{1}{2}k_1x^2 + \frac{1}{2}k_2x^2 = \frac{1}{2}kx^2$$

$$k_1 + k_2 = k$$

The units check. The equation is also reasonable. Now let's perform a reduction starting with the elemental equation:

$$F = kx$$

Two unwanted variables, F and x. Start substituting using the equation list.

$$F_1 + F_2 = kx$$

$$k_1x_1 + k_2x_2 = kx$$

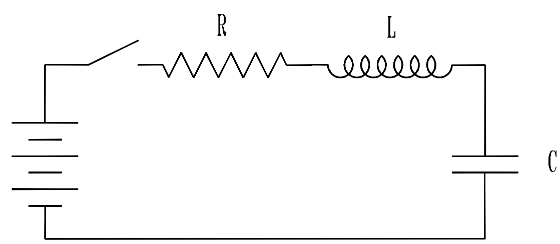
$$k_1x + k_2x = kx$$

$$k_1 + k_2 = k$$

We get the same expression for the equivalent spring rate starting with either the elemental equation or the energy equation. In general, the reduction is often easier starting with the energy equation.

Example 2: System Dynamics Problem

An electric circuit is shown in the schematic below. Derive a differential equation for the voltage across the capacitor.



Step 1: Draw and annotate a picture.

Electrical schematics are lumped parameter models of electric circuits, so we do not have to draw a picture. However, we do have to annotate the picture to show **ALL** of the variables and parameters used in our equations.

We must show an assumed direction for the current flow through each of the elements. The direction is arbitrary, analogous to the assumed direction of the force in a truss analysis. If the current actually flows in the direction opposite of what we assumed, the current will be negative. We must also identify which element the current flows through. The current will be identified with elemental subscripts. The current through resistor R1 will be identified as i_{R1}. The exception will be the current flow through a source which will not be given a subscript, unless there is more than one source in the system.

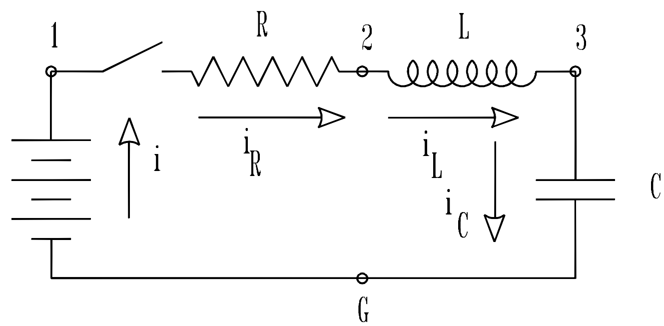
There are two ways to denote the potential (voltage) drop through an electric element, such as a resistor. One can either use a subscript(s) which indicates the element across which the voltage drops, e.g. V_{R1}, or two subscripts to indicate the **NODES** of distinct voltages between which the element is connected. We will use the node notation in ME 352 because it leads to fewer variables when elements are connected in parallel and it reduces sign errors. By convention, the order of the subscripts is the order of the subtraction to calculate the potential difference. e.g.:

$$v_1 - v_2 \equiv v_{12} \qquad \text{and} \qquad v_2 - v_1 \equiv v_{21}$$

Consequently, reversing the order of the subscripts inverts the sign of the voltage drop:

$$v_{12} = -v_{21} \quad \text{because} \quad v_1 - v_2 = -(v_2 - v_1)$$

The annotated schematic showing the nodes of distinct values of potential and the assumed positive direction for current through each element is:



Step 2. Write mathematical statements of physical truth.

1. **Compatibility Equations (also called Loop Equations):** These equations express the fact that voltage drops around loops must sum to zero. It makes no difference where you start the summation. It does make a difference how many compatibility equations you write because we only want independent equations. We must have as many compatibility equations as there are “internal” loops in the circuit. Internal loops are the smallest loops; the holes in the network. If we were mistakenly to write more compatibility equations than we have internal loops, we would generate dependent equations. Dependent equations are equations that can be derived from equations we already have. Dependent equations give you the familiar, but useless, result of 0 = 0.

There is one internal loop in this circuit. Using nodal subscripts, the compatibility equation is:

$$v_{g1} + v_{12} + v_{23} + v_{3g} = 0$$

2. **Continuity Equations (also called Node Equations):** These equations express the fact that, because nodes are geometric locations and not physical elements, and hence have no physical properties, nodes can not store electric charge. Current that flows in to a node must flow out. We will use the "checkbook sign convention" and define flow into a node to be positive and flow out of a node to be negative. We need one less node equation than we have nodes to avoid creating a dependent equation. By convention, we will not write a node equation for the ground node. The node equations for this simple circuit do not convey much information since the current is the same through each element. In the general case, however, the current though each element is unique.

Node 1: $i - i_R = 0$

Node 2: $i_R - i_L = 0$

Node 3: $i_L - i_C = 0$

3. **Elemental Equations:** DO NOT PANIC. You will not be required to remember the elemental equations for the many elements we will use in ME 352. The Official ME 352 Crib Sheet will be distributed with all of the exams. The only elemental equations you will be required to derive will be for some transformers and transducers, as will be discussed in lecture.

The elemental equation must respect the assumed positive direction of the current in the element indicated on the schematic. The potential (voltage) drop in the elemental equation is in the direction of the assumed current.

Resistor: $v_{12} = Ri_R$

Inductor: $v_{23} = L\frac{di_L}{dt}$

Capacitor: $i_C = C\frac{dv_{3g}}{dt}$

4. **Energy and Power Equations:** These are also given on the Official ME 352 Crib Sheet. There are two energy storage elements in this circuit, the capacitor and the inductor, and one dissipative element, the resistor. In ME 352, we will only need to write the equations for the energy storage elements, not the dissipative elements. The energy equations provide the initial conditions we need to solve the system equation. As we will discuss in lecture, the presence of two "independent" energy storage elements allows this circuit to oscillate, depending on the relative magnitude of the parameters. As we will also discuss, the energy storage variables will be our "state variables".

Inductor: $\mathbf{E}_L = \frac{1}{2}Li_L^2$

Capacitor: $\mathbf{E}_C = \frac{1}{2}Cv_{3g}^2$

3. Reduction and Solution.

The only difference between the reductions for the previous example of the mechanical structure and this circuit is that the elemental equations for the capacitor and inductor are ordinary differential equations with constant coefficients. Consequently, the system equation will also be an ordinary differential equation with constant coefficients. The order of the system equation will be equal to the number of energy storage elements; two in this case.

There are three firm rules to follow during the reduction a system equation:

> **1. Do not use the energy equations to derive the system equation. They are only used to provide the initial conditions needed to solve the system equation.**
>
> **2. Never integrate during the reduction, only differentiate. The system equation is a differential equation, not an integral equation.**
>
> **3. If you reach the end of the reduction and you have not introduced all of the elemental parameters, then there is an error.**

Other than these two rules, the reduction technique is identical to what we did above. Randomly pick any but a trivial equation and start substituting to eliminate all unwanted variables. Our input variable is the voltage across the battery, v_{1g}, and our output variable is the voltage across the capacitor, v_{3g}. It is generally easiest to start with an equation that contains either the input or the output variable. Let's start with the compatibility equation:

$$v_{g1} + v_{12} + v_{23} + v_{3g} = 0$$

Our input voltage is v_{1g}, not v_{g1}, since sources raise potentials above ground. Reversing the order of the subscripts on the source yields:

$$v_{1g} = v_{12} + v_{23} + v_{3g}$$

We must eliminate v_{12} and v_{23}. We can substitute the resistor elemental equation for v_{12}:

$$v_{1g} = Ri_R + v_{23} + v_{3g}$$

Now we must eliminate i_R. We use the node equations to express i_R as i_C:

$$i_R - i_L = 0 \qquad \text{rearranges as} \qquad i_R = i_L$$

and:

$$i_L - i_C = 0 \quad \text{rearranges as} \quad i_L = i_C$$

yielding:

$$i_R = i_C$$

Hence:

$$v_{1g} = Ri_C + v_{23} + v_{3g}$$

We can now use the capacitor elemental equation:

$$i_C = C\frac{dv_{3g}}{dt}$$

to eliminate i_C.

$$v_{1g} = RC\frac{dv_{3g}}{dt} + v_{23} + v_{3g}$$

v_{23} can be eliminated using the inductor elemental equation:

$$v_{23} = L\frac{di_L}{dt}$$

yielding

$$v_{1g} = RC\frac{dv_{3g}}{dt} + L\frac{di_L}{dt} + v_{3g}$$

or

$$v_{1g} = RC\frac{dv_{3g}}{dt} + L\frac{di_C}{dt} + v_{3g}$$

We appear to have a problem. How do we eliminate $\frac{di_C}{dt}$? Differentiate the capacitor elemental equation to yield:

$$\frac{di_C}{dt} = C\frac{d^2 v_{3g}}{dt^2}$$

and then substitute

$$v_{1g} = RC\frac{dv_{3g}}{dt} + LC\frac{d^2 v_{3g}}{dt^2} + v_{3g}$$

This is our system equation. Put it into standard form for higher order differential equations by arranging the terms in order of decreasing differentiation:

$$v_{1g} = LC\frac{d^2 v_{3g}}{dt^2} + RC\frac{dv_{3g}}{dt} + v_{3g}$$

We will not solve second order system equations for a while, but we can at least check the units. Remember, the equation can still be wrong, even if the units check.

What are the units of the derivative operator $\frac{d}{dt}$? Answer that question by thinking back to your first calculus course when the derivative was introduced. The operator d stands for an infinitesimally small change. It has no units. Consequently, the units of $\frac{d}{dt}$ are $\frac{1}{t}$. Square brackets are used to denote "the units of", so the previous sentence can be expressed mathematically as:

$$\left[\frac{d}{dt}\right] = \left[\frac{1}{t}\right]$$

What are the units of the second derivative with respect to time?

$$\left[\frac{d^2}{dt^2}\right] = \left[\frac{1}{t^2}\right]$$

Now, back to the system equation.

$$\left[v_{1g}\right] = \left[LC\frac{d^2 v_{3g}}{dt^2} + RC\frac{dv_{3g}}{dt} + v_{3g}\right]$$

The right-hand side of this differential equation a summation of three terms.

$$\left[v_{1g}\right] = \left[LC\frac{d^2 v_{3g}}{dt^2}\right] + \left[RC\frac{dv_{3g}}{dt}\right] + \left[v_{3g}\right]$$

All of these terms must have the same units as the single term on the left-hand side. Second grade math applies to differential equations. You can not add apples and oranges. First, simplify the expression by dropping the subscripts on the variables and by evaluating the units of the derivatives.

$$[v] = \left[LC\frac{v}{t^2}\right] + \left[RC\frac{v}{t}\right] + [v]$$

We are left with an expression consisting of the variables v and t and the elemental parameters. Determine the units of the parameters R, L, and C by rearranging the elemental equations:

Resistor: $v = Ri$ can be written as $R = \frac{v}{i}$ Hence: $[R] = \left[\frac{v}{i}\right]$

Inductor: $v = L\frac{di}{dt}$ can be written as $L = v\frac{dt}{di}$ Hence $[L] = \left[v\frac{t}{i}\right]$

Capacitor: $i = C\frac{dv}{dt}$ can be written as $C = i\frac{dt}{dv}$ Hence $[C] = \left[i\frac{t}{v}\right]$

Substitute these units into:

$$[v] = \left[LC\frac{v}{t^2}\right] + \left[RC\frac{v}{t}\right] + [v]$$

to yield:

$$[v] = \left[v\frac{t}{i}i\frac{t}{v}\frac{v}{t^2}\right] + \left[\frac{v}{i}i\frac{t}{v}\frac{v}{t}\right] + [v]$$

Cancel terms:

$$[v] = [v] + [v] + [v]$$

The units check.

Conclusion

The two key steps in engineering analysis are to formulate a reasonable model and write a complete set of independent mathematical statements of physical truth. Once these steps are complete, successful reduction is only a matter of persistence.