

Some Recent Results on the Einstein Constraint Equations

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ON THE EXISTENCE AND STABILITY OF THE PENROSE COMPACTIFICATION. In the 1960's Penrose [P] proposed a model of isolated gravitational systems based on the conformal compactification of Minkowski space. As the model has had enormous influence on the study of gravitational radiation, one would like to establish stability results which yield new examples through perturbation. Friedrich attacked this problem by rewriting the Einstein equations to emphasize the conformal structure, and he obtained a semi-global stability result for Minkowski space: for hyperboloidal data suitably close to a given hyperboloidal data set in Minkowski space (intersecting future null infinity), the resulting solution of the initial-value problem for Einstein's vacuum equation admits a conformal compactification to the future [F1]; see also more recent work of Anderson and Chruściel [AC]. Thus, if one could control the asymptotics near infinity on an asymptotically flat initial data set, in such a manner that it will evolve to a spacetime with suitable hyperboloidal slices (to the future and the past), then one could invoke the stability result to evolve the data to a spacetime which possesses a smooth compactification. In fact, Cutler and Wald [CW] use this method to produce solutions of the Einstein-Maxwell field equations that admit a smooth compactification.

We state a version of linearization stability of the conformal compactification (in the vacuum case) in terms of the initial data [C3]. The proof involves a careful study of the construction in [C1] of perturbations of given asymptotically flat, scalar flat metrics to ones which are Schwarzschild near infinity (in the time-symmetric case, the constraint equations reduce to the vanishing of the scalar curvature). Recall that at the flat metric, the linearization L of the scalar curvature operator is given by $L(h) = -\Delta(\text{tr}h) + \text{div}(\text{div}(h))$. The Euclidean metric is a critical point for the ADM mass function, in an appropriate space of solutions to the Einstein constraints ([B1], [B2], [CM]). We say that a solution h of $L(h) = 0$ is *nondegenerate* if the second variation of the mass in the direction of h is positive.

Theorem 1. *Let h be any smooth, compactly supported, symmetric $(0,2)$ -tensor on \mathbb{R}^3 with $L(h) = 0$, and for sufficiently small ϵ , let $g_\epsilon = u_\epsilon^4(\delta + \epsilon h)$ be asymptotically flat with zero scalar curvature. If h is nondegenerate, there is an $R_0 > 0$ so that for all ϵ small enough, there is a metric \bar{g}_ϵ of zero scalar curvature which agrees with g_ϵ in $\{x : |x| \leq R_0\}$ and is exactly Schwarzschild outside $\{x : |x| \geq 2R_0\}$, and so that the maximal Ricci-flat spacetime with the three-geometry \bar{g}_ϵ as a totally geodesic Cauchy surface admits a smooth conformal compactification. Moreover the path \bar{g}_ϵ is tangent to h at $\epsilon = 0$.*

We remark that one can approximate any solution h (in an appropriate weighted function space) of the linearized constraint $L(h) = 0$ by a compactly supported solution [C3]. Note that a TT-tensor (trace-free and divergence-free) with respect to the flat metric is in the kernel of L and is nondegenerate [BD]. It is known that there is an infinite-dimensional space of compactly supported TT-tensors at the flat metric ([Be], [DF], [C3]). We thus have as a corollary that there exists an

infinite-dimensional family of solutions of the vacuum constraint equations whose evolution admits a Penrose compactification; this echoes and augments the result of Chruściel and Delay [CD], who construct an infinite-dimensional family of such solutions which are parity-symmetric. We note that all of these constructed examples (including the Cutler-Wald examples) are Schwarzschild in a neighborhood of spatial infinity, which is consistent with the known restrictions at space-like infinity for asymptotically simple spacetimes as given by Friedrich [F2] and Valiente Kroon [VK].

ASYMPTOTICALLY FLAT AND SCALAR-FLAT METRICS ON \mathbb{R}^3 WITH MULTIPLE HORIZONS. We consider asymptotically flat initial data for the time-symmetric vacuum field equations, given by an asymptotically flat three-manifold (M, g) with zero scalar curvature. From Meeks, Simon and Yau [MSY], if M has nontrivial topology, then (M, g) has a stable minimal sphere (horizon). The natural question then is how to construct horizons on (\mathbb{R}^3, g) , where the topology is trivial. The first existence result was obtained by Beig and Ó Murchadha [BO] by conformally rescaling critical sequences of metrics for the conformal Laplacian on \mathbb{S}^3 . There is related work of Yan [Y] which gives metric criteria on (\mathbb{S}^3, g) to guarantee the existence of minimal spheres in the conformal rescaling G^4g on $\mathbb{S}^3 \setminus \{P\}$, where G is the Green's function at P of the conformal Laplacian. Further existence results have been obtained by Shi and Tam [ST], [S]. A construction due to Miao [Mi] produces examples by first filling in the Schwarzschild metric to produce a metric on \mathbb{R}^3 with nonnegative scalar curvature, and then using two types of scalar curvature deformation (one local and one conformal) to deform the metric to zero scalar curvature so that the horizon persists. One may apply Miao's construction to the multi-horizon data constructed by Chruściel-Delay [CD] (which has nontrivial topology) to prove the following theorem from [C2].

Theorem 2. *There exist asymptotically flat metrics on \mathbb{R}^3 with zero scalar curvature and multiple minimal spheres.*

The proof uses several methods of deforming the scalar curvature on a manifold: the conformal method, as well as two localized methods, one due to Lohkamp [L], and the other due to us [C1].

ON ISOPERIMETRIC SURFACES IN GENERAL RELATIVITY. One of the major recent developments in mathematical relativity is the resolution of the Riemannian case of the Penrose conjecture, by Huisken-Ilmanen [HI] and Bray [Br2]. Bray had obtained earlier partial results in his thesis [Br1] by using isoperimetric surface techniques. Bray established that the isoperimetric profile of the time-symmetric Schwarzschild initial data (of positive mass) is given by the radially symmetric spheres (*i.e.* these spheres are the surfaces homologous to the horizon which minimize area for net volume against the horizon), the method of proof of which has been codified in Bray-Morgan [BM]. The main idea is that one can deduce the isoperimetric profile of a given metric if one can construct an appropriate map to a model space (for instance Euclidean space or hyperbolic space) in which the

profile is known. We use the method to deduce the isoperimetric profile for the time-symmetric Reissner-Nordstrom and Schwarzschild-Anti-deSitter initial data [CGGK], which in each case is again given by the the radially symmetric spheres. In contrast, in the negative mass Schwarzschild, the radially symmetric spheres are unstable. For recently announced work by Huisken which explores the relation between isoperimetric inequalities and the mass of asymptotically flat metrics, see [H1], [H2].

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