ON ISOPERIMETRIC SURFACES IN GENERAL RELATIVITY

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Abstract. We obtain the isoperimetric profile for the standard initial slices in the Reissner-Nordstrom and Schwarzschild Anti-deSitter spacetimes, following recent work of Bray and Morgan on isoperimetric comparison. We then discuss these results in the context of Bray's isoperimetric approach to the Penrose inequality.

1. Introduction

One of the major recent developments in mathematical relativity is the resolution of the Riemannian case of the Penrose conjecture, by Huisken-Ilmanen [HI] and Bray [Br2]. Bray had obtained earlier partial results in his thesis [Br1] by using isoperimetric surface techniques. As a key step, Bray established that the isoperimetric profile of the time-symmetric Schwarzschild initial data (of positive mass) is given by the radially symmetric spheres, the method of proof of which has been codified in Bray-Morgan [BM]. The main idea is that one can deduce the isoperimetric profile of a given metric if one can construct an appropriate map to a model space (for instance, Euclidean space or hyperbolic space) in which the profile is known. We obtain below as a direct corollary of [BM] the isoperimetric profile for the Reissner-Nordstrom initial data. We then carry out an extension of the method to derive the isoperimetric profile for the Schwarzschild-Anti-deSitter (AdS) data; unlike the previous two families, which are asymptotically flat, Schwarzschild AdS is asymptotically hyperbolic. In all these cases, the spaces are rotationally symmetric, and the rotationally symmetric spheres give the isoperimetric profile. For contrast, in the negative mass Schwarzschild, the analogous family of spheres is unstable, as we discuss below.

We will review Bray's isoperimetric surface approach to the Penrose inequality and discuss its extension to certain asymptotically flat solutions of the Einstein-Maxwell constraint equations. We also include computations relevant to a form of the Penrose inequality for a class of asymptotically hyperbolic spaces. For background and references on the Penrose inequality, see [Br3], [BC], and for recently announced work by Huisken which explores the relation between isoperimetric inequalites and the mass of asymptotically flat metrics, see [H1], [H2].

2. Preliminaries

We recall the isoperimetric problem and introduce three families of metrics (Schwarzschild, Reissner-Nordstrom, and Schwarzschild AdS) whose isoperimetric profiles we will discuss.

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2.1. **Isoperimetric problems.** The isoperimetric problem is the classical problem of how to enclose a given volume V with a surface of least area. Note that in Euclidean and hyperbolic space, homogeneity and scaling allows one to conclude that if a volume V can be enclosed with a surface of area A, then a volume $V_0 < V$ can be enclosed with an area $A_0 < A$. In spaces without such scaling, one can pose an isoperimetric problem to find the minimum area that encloses a volume of at least V. It is a classical result that in Euclidean and hyperbolic spaces, the most efficient way to enclose a volume V is by using a sphere of the appropriate radius [Ch], [HHM]. We will in fact consider the problem of minimizing volume *against* a (two-sided) hypersurface Σ_0, *i.e.* we consider the problem of finding least-area enclosures in the homology class of Σ_0 of net volume (at least) V with Σ_0.

2.2. **The metrics of interest.** We will focus on three families of spherically symmetric metrics which appear as constant time slices in well-known solutions of the Einstein equations of general relativity. Let $\mathbb{S}^2$ be the two-dimensional sphere and let $d\Omega^2$ be the standard round metric on the unit two-sphere. Each of the following metrics is defined on the smooth manifold $(r_0, +\infty) \times \mathbb{S}^2$, where $m > 0$, and $r_0 > 0$ is specified below:

(1) Schwarzschild metric: $\left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$, $r_0 = 2m$.
(2) Schwarzschild AdS metric: $\left(1 + r^2 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$, where r_0 satisfies $1 + r_0^2 - \frac{2m}{r_0} = 0$.
(3) Reissner-Nordstrom metric: $\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$, where $m^2 > Q^2$ and r_0 is the larger solution of $1 - \frac{2m}{r_0} + \frac{Q^2}{r_0^2} = 0$.

The parameter m measures the deviation (in the third example, the top-order deviation) of the metrics from the model Euclidean or hyperbolic metrics. This parameter is called the *mass*, and indeed it has an interpretation in terms of the energy of isolated gravitational systems [Br3], [BC]. Please see the Appendix for useful formulas (Christoffel symbols, curvatures) for these metrics. Each metric extends to r_0, where there is a minimal sphere (*horizon*) S_{r_0}. This minimal sphere is in fact totally geodesic, and the metrics can be smoothly reflected across the horizon (using inversion in the horizon sphere with respect to the metric distance along radial geodesics) to produce complete metrics with two ends. These metrics are conformally flat; for example, the extended Schwarzschild metric in appropriate coordinates is precisely $\left(1 + \frac{m}{2r}\right)^4 \delta$, where δ is the Euclidean metric. In these coordinates the horizon is located at $r = m/2$, and the inversion $r \mapsto \frac{m^2}{4r}$ is an isometry.

We note that the Schwarzschild metric with $m > 0$ can be isometrically embedded into the Euclidean space $\mathbb{R}^4$ as the set $\{(x, y, z, w) : r = \frac{w^2}{8m} + 2m\}$, where $r^2 = x^2 + y^2 + z^2$. To see this, we look for an embedding which in terms of spherical coordinates on Schwarzschild is of the form $(r, \omega) \mapsto (r\omega, \xi(r)) \in \mathbb{R}^4$. Using the above form of the Schwarzschild metric, we see that the map is an isometry if and only if $(\xi'(r))^2 + 1 = \left(1 - \frac{2m}{r}\right)^{-1}$, which can be rewritten (choosing $\xi'(r) > 0$) as $\xi'(r) = \sqrt{\frac{2m}{r-2m}}$. We note that $\xi(r) = \sqrt{8m(r - 2m)}$ does indeed satisfy this equation. Interestingly enough, this derivation breaks down for $m < 0$; however (as was pointed out to us by Greg Galloway and Hubert Bray), it is also not hard to see that the same idea can be pushed through in the negative mass case to obtain an isometric embedding of the negative mass Schwarzchild into *Minkowski* space.

2.3. **The Einstein constraint equations.** The three families of metrics above give particular solutions to the Einstein constraint equations, as we now recall. The Einstein equations for the corresponding four-dimensional Lorentzian spacetimes $(\mathcal{S}, \bar{g})$ in which these three-dimensional Riemannian spaces embed as totally geodesic spacelike slices are, respectively, $Ric(\bar{g}) = 0$, $Ric(\bar{g}) = -3\bar{g}$, and $Ric(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} = 8\pi T$, where T is the stress-energy tensor of a Maxwell field [W]. Consider in general any spacetime $(\mathcal{S}, \bar{g})$ satisfying one of these Einstein equations; then the Gauss and Codazzi equations (together with the Einstein equation) imply constraint equations on the geometry (intrinsic and extrinsic) of spacelike slices. If g is the induced metric and $I\!I$ the second fundamental form (with trace H) of a spacelike slice, then using the Einstein equation along with the Gauss equation, we obtain the Hamiltonian constraint, which in the first two cases yields $R(g) - \|I\!I\|^2 + H^2 = 0$, respectively $R(g) - \|I\!I\|^2 + H^2 = -6$. In the totally geodesic ($I\!I = 0$) case, these constraints reduce to the condition of constant scalar curvature $R(g) = 0$ or $R(g) = -6$, respectively; in the case of a maximal slice ($H = 0$), the constraints imply the respective inequalities $R(g) \geq 0$ and $R(g) \geq -6$. Similarly, the (totally geodesic) Einstein-Maxwell constraint equations for a metric g and an electromagnetic field E are given by the Hamiltonian constraint $R(g) = 2|E|^2$ coupled with the Maxwell field equation $div_g E = 0$. If we let $\mathbf{e}_r$ be the unit outward radial vector, and couple the field $E = \frac{Q}{r^2}\mathbf{e}_r$ to the Reissner-Nordstrom metric, we produce a solution to the Einstein-Maxwell constraints.

2.4. **On the isoperimetric inequality and the mass.** In Euclidean space, the isoperimetric inequality for a closed surface Σ of area A enclosing a volume V can be written $V \leq \frac{A^{3/2}}{6\sqrt{\pi}}$, with equality precisely in the case Σ is a round sphere. We compare this to Schwarzschild, where (using Cor. 3.2) it is easy to compute the volume $V(\sigma)$ enclosed by the isoperimetric sphere of area σ. In fact, if we use the conformally flat coordinates for Schwarzschild, in which the metric is $(1+\frac{m}{2r})^4\delta$, then we have $A(S_r) = 4\pi r^2(1+\frac{m}{2r})^4$. Thus we have $\frac{(A(S_r))^{3/2}}{6\sqrt{\pi}} = \frac{4\pi}{3}r^3\big(1+\frac{3m}{r}+mO(\frac{1}{r^2})\big)$. Furthermore, the net volume enclosed by S_r has the expansion

$$4\pi \int_{m/2}^{r} (1 + \tfrac{m}{2t})^6 \, t^2 \, dt = \tfrac{4\pi}{3}r^3\big(1 + \frac{9m}{2r} + mO(\tfrac{1}{r^2})\big).$$

From this it is easy to see that the volume enclosed by the isoperimetric sphere of area σ has the expansion $V(\sigma) = \frac{\sigma^{3/2}}{6\pi^{1/2}}\big(1 + \frac{(3\sqrt{\pi})m}{\sqrt{\sigma}} + mO(\frac{1}{\sigma})\big)$. This is yet another quantitative way in which the mass m measures the deviation of the geometry from that of Euclidean, which is explored in the recent work of Huisken [H1], [H2].

3. Isoperimetric Profiles by Comparison

In this section we will review the isoperimetric comparison theorem of Bray and Morgan and apply it to the Schwarzschild and Reissner-Nordstrom spaces. Let $I \subset \mathbb{R}$ be an interval. Suppose we have a rotationally symmetric model space $M_0 = I \times \mathbb{S}^2$ with the twisted product metric $dr^2 + \varphi_0^2(r)d\Omega^2$ for which we know the isoperimetric surfaces are the radially symmetric spheres $S_c = \{r = c\}$. We consider another rotationally symmetric space $M = I \times \mathbb{S}^2$ with the metric $dr^2 + \varphi^2(r)d\Omega^2$. Bray and Morgan showed that under certain geometric conditions, the isoperimetric surfaces in M are also the radially symmetric spheres. We now recall their argument, which as in [BM] can be more generally applied to twisted products $I \times N$ with a closed manifold fiber N.

Let $F : M \to M_0$ map radially symmetric spheres in M to radially symmetric spheres in M_0, so that $F(r, \omega) = (\psi(r), \omega)$. We assume that ψ is increasing, so that F is orientation-preserving. We define the area stretch AS_Σ for a surface $\Sigma \subset M$ by the equation $F^*(dA_{F(\Sigma)}) = AS_\Sigma \cdot dA_\Sigma$, where dA_Σ and $dA_{F(\Sigma)}$ are the area forms of $\Sigma \subset M$ and $F(\Sigma) \subset M_0$, respectively. The volume stretch VS is defined similarly by $F^*(dV_{M_0}) = VS \cdot dV_M$, where dV_M and dV_{M_0} are the volume forms of M and M_0, respectively. By symmetry, VS depends only on r. Finally, let $A(\Sigma)$ be the area of the surface $\Sigma \subset M$, and let $A_0(\Sigma_0)$ be the area of the surface $\Sigma_0 \subset M_0$.

Let $a = A(S_{r_1})/A_0(F(S_{r_1}))$. Suppose the map F can be constructed so that the area stretch under F satisfies $AS_\Sigma \leq \frac{1}{a}$, and so the volume stretch satisfies $VS(r) \leq b$ for $r < r_1$, and $VS(r) \geq b$ for $r > r_1$. Now suppose there were a surface $\Sigma \subset M$ bounding nonnegative net volume against S_{r_1} (*i.e.* Σ bounds no less volume against S_{r_0} as S_{r_1} does), so that Σ has the same or less surface area as S_{r_1}. We will show that in fact $A(\Sigma) = A(S_{r_1})$, which will then imply that S_{r_1} is an isoperimetric surface. Since the volume stretch for $r > r_1$ is no less than the volume stretch for $r < r_1$, the net volume bounded by $F(\Sigma)$ contained in $\{r > \psi(r_1)\}$ is no less than the net volume bounded by $F(\Sigma)$ contained in $\{r < \psi(r_1)\}$. Thus the net volume bounded by $F(\Sigma)$ is greater than or equal to the volume bounded by $F(S_{r_1})$. Since the area stretch $AS_\Sigma \leq \frac{1}{a} = A_0(F(S_{r_1}))/A(S_{r_1})$, and $A(\Sigma) \leq A(S_{r_1})$, we obtain

$$\begin{aligned} A_0(F(\Sigma)) &= \int_{F(\Sigma)} dA_{F(\Sigma)} = \int_\Sigma F^* dA_{F(\Sigma)} = \int_\Sigma AS_\Sigma \cdot dA_\Sigma \\ &\leq \tfrac{1}{a} \cdot A(\Sigma) = A_0(F(S_{r_1})) \frac{A(\Sigma)}{A(S_{r_1})} \leq A_0(F(S_{r_1})). \end{aligned}$$

Since $F(S_{r_1}) = S_{\psi(r_1)}$ is an isoperimetric surface in M_0, $F(\Sigma)$ and $F(S_{r_1})$ must thus bound the same amount of volume and have the same surface areas, $A_0(F(\Sigma)) = A_0(F(S_{r_1}))$. Thus the above inequalities must be equalities, and so we see that indeed $A(\Sigma) = A(S_{r_1})$. Therefore we have shown by comparison that S_{r_1} is an isoperimetric surface in M; if we have uniqueness for the isoperimetric surfaces, we can go further to assert $\Sigma = S_{r_1}$.

To put this observation to work, one identifies concrete geometric conditions that allow such a map F to be constructed. Indeed the main theorem in [BM] is stated in geometric terms from which the following is readily established as a corollary. We note that the comparison space M_0 for this corollary is Euclidean space, so the comparison metric is $dr^2 + r^2 d\Omega^2$.

Theorem 3.1. [BM] *Consider a rotationally symmetric three-manifold* $M = I \times \mathbb{S}^2$ *with the metric* $dr^2 + \varphi^2(r)d\Omega^2$. *Suppose (1)* φ' *is nondecreasing for all* r, *and (2)* $0 \leq \varphi' \leq 1$ *for all* $r \geq r_0$. *Then for all* $r \geq r_0$, *the radially symmetric spheres* S_r *minimize surface area among smooth surfaces enclosing the same volume with* S_{r_0}, *where volume inside* $\{r < r_0\}$ *is counted as negative. Furthermore, these spheres are unique minimizers if* $\varphi'(r) < 1$.

Condition (1) holds if and only if M *has nonpositive radial Ricci curvature. For any* r, *condition (2) holds if and only if* S_r *has nonnegative (inward) mean* *curvature and M has nonnegative tangential sectional curvature, or equivalently, S_r has nonnegative Hawking mass.*

We take the mean curvature to be the trace of the second fundamental form (the sum of the principal curvatures), not the average of the principal curvatures (as in [BM]). We recall the Hawking mass of a surface Σ is $m_H(\Sigma) = \sqrt{\frac{A(\Sigma)}{16\pi}}\Big(1 - \frac{1}{16\pi}\int_\Sigma H^2\, dA\Big)$. We will see the Hawking mass play a role in the Penrose inequality below; in fact, the underlying motivation for the Huisken-Ilmanen inverse mean curvature flow is the monotonicity of the Hawking mass under the flow [G].

We are interested in spaces (M, g) with $M = I \times \mathbb{S}^2$ and with $g = f(r)dr^2 + r^2 d\Omega^2$, where f is a positive function. The metrics which we study here, in both the forms given above and for the metrics suitably extended by reflection, all have this form. In order to apply Theorem 3.1 to such spaces, we note that we can express g as a twisted product metric.

Lemma 3.1. *The metric g can be written as a twisted product metric $g = dt^2 + \varphi^2(t)d\Omega^2$, where $\varphi(t) > 0$.*

Proof. The desired result is equivalent to $dt = \sqrt{f(r)}\, dr$, for $r = \varphi(t)$. We integrate to find $t = t(r)$; by the equation t is increasing, and we write the inverse as $r = \varphi(t)$. □

Since g can be written as a twisted product metric, we can apply the Theorem 3.1 provided f satisfies certain conditions.

Theorem 3.2. *Suppose $f'(r) \leq 0$ for all r and $f(r) \geq 1$ for $r \geq r_0$. Then every sphere of revolution S_r for $r \geq r_0$ minimizes perimeter among smooth surfaces enclosing fixed volume with S_{r_0}, uniquely if $f(r) > 1$ for $r \geq r_0$.*

Proof. It suffices to show that M satisfies the conditions for Theorem 3.1, in particular that M has nonpositive radial Ricci curvature, S_r has nonnegative mean curvature (with respect to the inward unit normal), and M has nonnegative tangential sectional curvature. For indices, let $(1, 2, 3)$ represent (r,ϕ,θ). We find $R_{12} = R_{13} = 0$ and $R_{11} = \frac{f'}{rf}$ (see the Appendix), so the radial Ricci curvature is nonpositive if and only if $f' \leq 0$. We know $H_{S_r} = \frac{2}{r\sqrt{f}} > 0$ as required. We compute the sectional curvature K of the plane containing ∂_ϕ and ∂_θ as

$$K = \frac{g_{33}R^3_{232}}{g_{22}g_{33} - (g_{23})^2} = \frac{\left(1 - \frac{1}{f(r)}\right) r^2 \sin^2\phi}{r^2 \cdot r^2 \sin^2\phi - 0^2} = r^{-2}\left(1 - \frac{1}{f(r)}\right)$$

Thus $K \geq 0$ if and only if $f \geq 1$. The spheres S_r are uniquely minimizing provided $f > 1$. □

Remark 3.3. It is often convenient to consider the function $1/f$ instead of f. If $h = 1/f$, $f' = -h'/h^2$, so $f' \leq 0$ if and only if $h' \geq 0$. To check if $f \geq 1$, we check if $h \leq 1$ and similarly for strict inequality, in which case the tangential sectional curvature is strictly positive.

3.1. **The Schwarzschild profile.**

We let g be the Schwarzschild metric with $m > 0$, which we recall has the form $(1 - \frac{2m}{r})^{-1}dr^2 + r^2 d\Omega^2$ on $(2m, +\infty) \times \mathbb{S}^2$. We recall the following result from [Br1], proved as in [BM].

Corollary 3.2. [Br1] *In the Schwarzschild metric with positive mass $m > 0$, every sphere of revolution S_r for $r \geq 2m$ uniquely minimizes perimeter among smooth surfaces enclosing fixed volume with the horizon S_{2m}.*

Proof. Let $h(r) = 1 - \frac{2m}{r}$. We note $h(r) = 1 - \frac{2m}{r} < 1$ for positive mass. Also, $h'(r) = \frac{2m}{r^2} > 0$, so by Theorem 3.2, every sphere of revolution S_r for $r \geq 2m$ uniquely minimizes perimeter among smooth surfaces enclosing fixed volume with the horizon S_{2m}. ☐

Remark 3.4. Of course if we consider the full Schwarzschild space with reflection symmetry, then uniqueness is with respect to one chosen end. Similar considerations apply to Reissner-Nordstrom and Schwarzschild AdS below.

3.2. **The Reissner-Nordstrom profile.** Let g be the Reissner-Nordstrom metric, which on $(r_0, \infty) \times \mathbb{S}^2$ takes the form $g = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$. We shall assume $m^2 > Q^2$ so that $h(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}$ has two positive roots, and we take r_0 to be the larger of the two roots. Then it is easy to see $r_0 > m > \frac{Q^2}{m}$.

Corollary 3.3. *In Reissner-Nordstrom with $m^2 > Q^2$, every sphere of revolution S_r for $r \geq r_0$ uniquely minimizes perimeter among smooth surfaces enclosing fixed volume with S_{r_0}.*

Proof. We have $h(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} < 1$ for $r > \frac{Q^2}{2m}$. We also have $h'(r) = \frac{2m}{r^2} - \frac{2Q^2}{r^3}$ so that $h'(r) \geq 0$ for $r \geq \frac{Q^2}{m}$. Both conditions of Theorem 3.2 hold for $r \geq r_0$, so every sphere of revolution S_r for $r \geq r_0$ uniquely minimizes perimeter among smooth surfaces enclosing fixed volume with S_{r_0}. ☐

4. ISOPERIMETRIC PROFILE FOR SCHWARZSCHILD ADS

We now let the comparison space M_0 be hyperbolic three-space with hyperbolic metric $(1 + r^2)^{-1} dr^2 + r^2 d\Omega^2$. Consider $M = (r_0, \infty) \times \mathbb{S}^2$ with the Schwarzschild AdS metric $g = \left(1 + r^2 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$. We will construct a comparison map $F : M \to M_0$ given by $F(r, \omega) = (\psi(r), \omega)$ to show that the radially symmetric spheres are the isoperimetric surfaces in Schwarzschild AdS space.

We will be concerned with two particular types of area stretches. The first one encodes the area stretch for a radially symmetric sphere, $F^*(dA_{F(S_r)}) = AS_1 \cdot dA_{S_r}$:

$$AS_1(r) = \frac{\int_{\mathbb{S}^2} \psi^2(r) dA_{\mathbb{S}^2}}{\int_{\mathbb{S}^2} r^2 dA_{\mathbb{S}^2}} = \frac{\psi^2(r)}{r^2}.$$

For example, in the previous section, we had $AS_1 = \frac{1}{a}$.

The second stretch factor encodes the area stretch for an annular surface $\Sigma = J \times \mathbb{S}^1$ ($J \subset (r_0, +\infty)$) which is obtained by flowing some great circle $\mathbb{S}^1$ (with element of arclength ds) along the radial direction field ∂_r, $F^*(dA_{F(\Sigma)}) = AS_2 \cdot dA_\Sigma$:

$$AS_2(r) = \frac{\frac{d}{dr} \int_{\psi(r_0)}^{\psi(r)} \int_{\mathbb{S}^1} \rho(1 + \rho^2)^{-1/2} ds\, d\rho}{\frac{d}{dr} \int_{r_0}^{r} \int_{\mathbb{S}^1} \rho(1 + \rho^2 - 2m/\rho)^{-1/2} ds\, d\rho} = \frac{\psi(r)(1 + \psi^2(r))^{-1/2} \psi'(r)}{r(1 + r^2 - 2m/r)^{-1/2}}.$$

The volume stretch VS, where $F^*(dV_{M_0}) = VS \cdot dV_M$, is given by

$$\begin{aligned} VS(r) &= \frac{\frac{d}{dr}\int_{\psi(r_0)}^{\psi(r)} \int_{\mathbb{S}^2} \rho(1+\rho^2)^{-1/2} dA_{\mathbb{S}^2} d\rho}{\frac{d}{dr}\int_{r_0}^{r} \int_{\mathbb{S}^2} \rho^2(1+\rho^2-2m/\rho)^{-1/2} dA_{\mathbb{S}^2} d\rho} \\ &= \frac{\psi^2(r)(1+\psi^2(r))^{-1/2}\psi'(r)}{r^2(1+r^2-2m/r)^{-1/2}}. \end{aligned}$$

Note that $VS = \sqrt{AS_1}AS_2$.

Lemma 4.1. *The area stretch AS_Σ for any surface does not exceed the maximum of AS_1 and AS_2.*

Proof. If Σ is any smooth surface, then by dimension considerations, $T_p\Sigma$ contains at least one tangent direction to the radial sphere through p. We let E_1 be such a unit vector; let E_2 be an orthogonal unit vector tangent to the radial sphere, and let E_3 be the unit outward radial vector. There exist α and β with $\alpha^2 + \beta^2 = 1$ so that $\alpha E_2 + \beta E_3 \in T_p\Sigma$. We have by orthogonality

$$\begin{aligned} AS_\Sigma &= dA_{F(\Sigma)}(F_*(E_1), \alpha F_*(E_2) + \beta F_*(E_3)) \\ &= |F_*(E_1)||\alpha F_*(E_2) + \beta F_*(E_3)| \\ &= \frac{\psi(r)}{r}\sqrt{\alpha^2\frac{\psi(r)^2}{r^2} + \beta^2\frac{(\psi'(r))^2((1+\psi(r))^{-1}}{(1+r^2-2m/r)^{-1}}}. \end{aligned}$$

Thus we have $AS_\Sigma^2 = \alpha^2 AS_1^2 + \beta^2 AS_2^2$, from which the claim follows. ☐

As above, we will produce a map $F : M \to M_0$ with the following properties: at $r = r_1$, the area stretch $AS_1(r_1) = \frac{1}{a}$ and the volume stretch $VS(r_1) = b$, for some $a, b > 0$; for all Σ, $AS_\Sigma \leq \frac{1}{a}$; $VS(r) \geq b$ for $r > r_1$, and $VS(r) \leq b$ for $r < r_1$. We note that by the lemma, it suffices to show that $AS_1, AS_2 \leq \frac{1}{a}$ everywhere. We also note that the construction in [BM] uses only the parameter a, in which case $VS(r_1) = \frac{1}{a}$; this suffices for the asymptotically flat cases above, but we require slightly more flexibility in constructing the map F in the asymptotically hyperbolic case, and so we introduce the parameter b. As above, we have as a consequence that for any competitor surface Σ bounding at least as much volume as S_{r_1} with equal or less surface area, the image $F(\Sigma)$ will bound no less volume with no more surface area than $F(S_{r_1})$. (We note that in hyperbolic space M_0, we can shrink $F(\Sigma)$ to produce a surface Σ' bounding the same volume as $F(S_{r_1})$ with less or equal surface area.) But the radially symmetric spheres are isoperimetric surfaces in hyperbolic space M_0, so all previously mentioned area and volume inequalities must be equalities, and hence radially symmetric spheres are isoperimetric surfaces in M. Furthermore, if the maximal area stretch is strictly tangential (*i.e.* $AS_2 < \frac{1}{a}$), then the radially symmetric spheres are the unique isoperimetric surfaces in M.

Theorem 4.1. *In Schwarzschild AdS, every sphere of revolution S_r uniquely minimizes perimeter among smooth surfaces enclosing fixed volume with S_{r_0}.*

Proof. First we consider $r_1 > 2m$. Let $a = 1 - 2m/r_1 < 1$, and define F using $\psi(r) = a^{-1/2}r$ for all $r \geq r_0$. Then $AS_1 = \frac{1}{a}$ everywhere. Also

$$AS_2(r) = \frac{\sqrt{1+r^2-2m/r}}{a\sqrt{1+a^{-1}r^2}}.$$

Hence $AS_2(r) < \frac{1}{a}$ is equivalent to $1 - 2mr^{-3} < \frac{1}{a}$. Since $1 - 2mr^{-3} < 1 < \frac{1}{a}$ for all $r > 0$, the maximal area stretch equals $AS_1 = \frac{1}{a}$, is strictly tangential, and occurs on S_{r_1}.

Now note that at $r = r_1$, $AS_2(r_1) = a^{-1/2}$, and as $r \to \infty$, $AS_2(r) \to a^{-1/2}$. We have

$$\begin{aligned}\frac{d}{dr}\left(\frac{1 + r^2 - 2m/r}{1 + a^{-1}r^2}\right) &= \frac{(2r + 2m/r^2)(1 + a^{-1}r^2) - (1 + r^2 - 2m/r)(2a^{-1}r)}{(1 + a^{-1}r^2)^2} \\ &= \frac{2(a-1)r^3 + 6mr^2 + 2am}{ar^2(1 + a^{-1}r^2)^2}.\end{aligned}$$

The cubic numerator has a positive local minimum at $r = 0$ and one other critical point at some $r > 0$, so in particular it has only one root (which is positive). Thus AS_2 has a unique maximum on the set $r \geq r_0$. Since AS_2 decreases to $a^{-1/2}$ as $r \to \infty$, the maximum occurs on (r_1, ∞), and on this interval $AS_2(r) > AS_2(r_1) = a^{-1/2}$. Hence $VS(r) = \sqrt{AS_1(r)}AS_2(r) \leq a^{-1/2} \cdot a^{-1/2} = 1/a$ for $r \leq r_1$, and $VS(r) = \sqrt{AS_1(r)}AS_2(r) \geq a^{-1/2} \cdot a^{-1/2} = \frac{1}{a}$ for $r \geq r_1$. Since areas and volumes stretch in the required manner, S_r are the unique isoperimetric surfaces for $r > 2m$.

Now suppose $r_1 \leq 2m$. Choose $a \in (0, 1)$ and let $\psi(r) = a^{-1/2}r$ for all $r \geq r_0$. Then $AS_1 = \frac{1}{a}$ and $AS_2 < \frac{1}{a}$ everywhere as before. Note that at $r = r_1$, $AS_2(r_1) < a^{-1/2}$ since $1 - 2m/r_1 \leq 0$. As before, AS_2 has a unique maximum for $r \geq r_0$ and AS_2 decreases to $a^{-1/2}$ as $r \to \infty$. Hence the maximum occurs for some $r_{max} > r_1$, and AS_2 is increasing on (r_0, r_{max}). Thus the volume stretch $VS = a^{-1/2}AS_2$ is also increasing on (r_0, r_{max}), and so $VS(r) \leq b := VS(r_1)$ for $r < r_1$, and $VS(r) \geq b$ for $r \in [r_1, r_{max}]$. Furthermore, $b = VS(r_1) = \sqrt{AS_1(r_1)}AS_2(r_1) < \frac{1}{a}$, so for $r > r_{max}$, $VS(r) = \sqrt{AS_1(r)}AS_2(r) \geq \frac{1}{a} > b$. Since areas and volumes stretch in the required manner, S_r are the unique isoperimetric surfaces for $r \leq 2m$. □

5. Remarks on the negative mass Schwarzschild

If we let the mass m be negative in the formula $(1 - \frac{2m}{r})^{-1}dr^2 + r^2 d\Omega^2$ for the Schwarzschild metric, we obtain an inextendible metric with no minimal sphere. The coordinates are only singular at the origin; in fact the metric is incomplete, as radial geodesics have finite length as $r \to 0^+$, but the Ricci tensor blows up on approach to the origin. The Bray-Morgan construction for the positive-mass Schwarzschild does not extend to the negative mass case; in fact we will show below that radial spheres are unstable.

5.1. **Instability of the radial spheres.** We consider now the variations of area and volume enclosed by the coordinate spheres, and in particular we compute the second variation of area with respect to volume-preserving perturbations. The variation formulas are standard [Ch], [T]. We note that H below is the trace of the second fundamental form computed with respect to the *inward* unit normal $-\nu$, which accounts for a sign difference from some versions of the variation formulas.

We consider a smooth family of surfaces Σ_t obtained from $\Sigma = \Sigma_0$ using the variation field given by $V(x,t) = \eta(x,t)\nu(x,t)$. Then we have the first variation $A'(t) = \int_{\Sigma_t} H\eta \, dA$, and the second variation

$$A''(0) = \int_{\Sigma} \left[\eta(-\Delta_\Sigma \eta - \eta \|I\!I\|^2 - \eta Ric(\nu,\nu)) + H\frac{\partial \eta}{\partial t} + H^2\eta^2\right] dA.$$

We also have that the first variation of volume $V(t)$ inside Σ_t is given by $V'(t) = \int_{\Sigma_t} \eta \, dA$, and so the second variation is $V''(t) = \int_{\Sigma_t} (H\eta^2 + \frac{\partial \eta}{\partial t}) dA$.

The radial spheres $\Sigma = S_r$ have constant mean curvature, and hence they are critical points for the area functional with respect to volume-preserving perturbations. Indeed, from the variation of volume formula, we have $0 = V'(0) = \int_\Sigma \eta \, dA$, which implies that $A'(0) = 0$ too. If we now consider the second variation *at* S_r, since the mean curvature H is *constant* we have $0 = HV''(0) = \int_{S_r} (H^2\eta^2 + H\frac{\partial \eta}{\partial t}) dA$. Thus the second variation formula simplifies; if we also apply the divergence theorem to the first term, we then have

$$A''(0) = \int_{S_r} \left[|\nabla^\Sigma \eta|^2 - \eta^2 \|I\!I\|^2 - \eta^2 Ric(\nu, \nu) \right] dA. \tag{5.1}$$

From the Appendix we have $\nu = \sqrt{1 - \frac{2m}{r}} \, \partial_r$, $Ric(\nu, \nu) = -\frac{2m}{r^3}$ and $\|I\!I\|^2 = \frac{2}{r^2}(1 - \frac{2m}{r})$. When we plug this into the preceding equation we get $A''(0) = \int_{S_r} \left[|\nabla^\Sigma \eta|^2 - \eta^2 \big(\frac{2}{r^2} \big(1 - \frac{3m}{r} \big) \big) \right] dA$.

It is well known [ABR] that the lowest non-zero eigenvalue λ_1 for the Laplacian on a round two-sphere $\mathbb{S}^2_\kappa$ of curvature κ is $\lambda_1 = 2\kappa$, with eigenspace spanned by the restriction of the coordinate functions x, y, z to the sphere (isometrically embedded in $\mathbb{R}^3$ centered at the origin): *e.g.* $\Delta_{\mathbb{S}^2_\kappa}(x) = -2\kappa x$. We now invoke the Poincaré inequality we obtain from the decomposition of $L^2(\Sigma)$ by the eigenspaces of the Laplacian [Ch]: $\lambda_1 \int_\Sigma \eta^2 \, dA \leq \int_\Sigma |\nabla^\Sigma \eta|^2 \, dA$, for all η with $\int_\Sigma \eta \, dA = 0$; equality holds precisely for functions in the λ_1-eigenspace. Applying this with $\Sigma = S_r$ we have $\lambda_1 = \frac{2}{r^2}$, so that

$$A''(0) \geq \int_{S_r} \frac{6m}{r^3} \eta^2 \, dA, \tag{5.2}$$

with equality if and only if η is in the λ_1-eigenspace. We see from this that in the *positive* mass Schwarzschild case, the second variation must be positive for (nontrivial) volume-preserving deformations (which we knew already from the isoperimetric profile) . But in the negative mass case, we see that for η a coordinate function, the right-hand side of Eq. (5.2) is negative. We note that $\eta(x, t) = x$ does not satisfy $V''(0) = 0$. To satisfy this condition, we can let $\eta_0(x, t) = x + \alpha t$, where α is a constant chosen precisely so that $V''(0) = 0$. Then η_0 generates a deformation that preserves volume to second order; from here it is not hard to modify the variation by a scaling to preserve volume, and so that the corresponding η has first-order Taylor expansion η_0. Another way to see that the spheres do not minimize area for a given volume is by considering the variation $\eta_0 \nu$. Since this variation leaves the volume unchanged to second-order in t, the change in volume is $O(t^3)$. Now, the volume $V(S_r)$ enclosed by the radial spheres satisfies $\frac{dV(S_r)}{dr} = \frac{4\pi r^2}{\sqrt{1 - \frac{2m}{r}}} > 0$, so we see that the radius $r(t)$ of the radial sphere with volume $V(t)$ is such that $(r(t) - r) = O(t^3)$. So the area $A(S_{r(t)}) = A(S_r) + O(t^3)$, and thus $A''(0) < 0$ implies that for some C and small $t > 0$, the area $A(t)$ of Σ_t satisfies $A(t) < A(S_r) - Ct^2 < A(S_{r(t)})$. This should not be surprising by considering the growth of the volume for small r:

$$V(S_r) = 4\pi \int_0^r t^2 \sqrt{\frac{1}{1 - \frac{2m}{t}}} \, dt < 4\pi \int_0^r t^2 \sqrt{\frac{t}{2|m|}} \, dt = O(r^{7/2}).$$

This volume growth is slower than for the Euclidean metric $dr^2 + r^2 d\Omega^2$, but the radial spheres have the same area as in the Euclidean metric, so that it is more efficient to slide them off-center. It might be interesting to consider the isoperimetric problem in this singular space, and whether optimizing shapes tend to singular varieties that go *through* the singular point.

6. THE PENROSE INEQUALITY FROM ISOPERIMETRIC TECHNIQUES

The Riemannian Penrose inequality is a lower bound on the ADM mass of an asymptotically flat metric of nonnegative scalar curvature in terms of the areas of certain horizons. There are a host of partial results, including the isoperimetric approach of [Br1], and then there are the proofs of [Br2], [HI]. We state the version from [Br2].

Theorem 6.1 (Penrose Inequality)**.** *Let (M, g) be asymptotically flat with $R(g) \geq 0$. Let m be the ADM mass of an end, and let A be the total surface area of the outermost minimal spheres with respect to this end. Then*

$$m \geq \sqrt{\frac{A}{16\pi}}.$$

There are various analogues of this inequality that are sought [BC], including asymptotically hyperbolic versions and versions with charge. We discuss an example each for both types, to illustrate that the beautiful arguments of Bray which connect the isoperimetric profiles to the Penrose inequality in [Br1] extend to the context of the isoperimetric profiles obtained above.

6.1. Variation of area along an isoperimetric profile.

We again consider the isoperimetric problem of minimizing area for volume V between a horizon and competitor surfaces in the homology class of the horizon. We assume we have an isoperimetric profile $\Sigma(V)$, each surface of which is connected. The objective in the next sections will be to establish that a mass function $m(V)$ associated with the Hawking mass function $m_H(\Sigma(V))$ determined by the isoperimetric profile is nondecreasing, for which we now derive a key inequality. We compute the variation of the area function $A(V)$ of the profile, where we employ the harmless abuse of notation, $A(V) := A(\Sigma(V))$, and we note that $A(0) = A(\Sigma_0)$. The area function of the isoperimetric profile may not be smooth in V, so that this fact is established in a weak but sufficient form. To be precise, for each $V_0 > 0$, we let $A_{V_0}(V)$ be the area of the surface obtained by flowing $\Sigma(V_0)$ in the outward normal direction at unit speed until the volume enclosed with the horizon is V. A_{V_0} will be smooth for V near V_0. Moreover, $A_{V_0}(V_0) = A(V_0)$ and $A_{V_0}(V) \geq A(V)$. Thus if A were smooth, then $A'(V_0) = A'_{V_0}(V_0)$ and $A''(V_0) \leq A''_{V_0}(V_0)$; so an inequality for the derivatives of A_{V_0} at V_0 can be interpreted as a weak (distributional) inequality for the derivatives of A. We let $\Sigma^t_{V_0}$ be the surface obtained by flowing $\Sigma(V_0)$ for time t, and let $V(t)$ be the volume this surfaces encloses with the horizon. Then by the equations of variation (as recalled in the preceding section), we have $\frac{d}{dt}(A_{V_0}(V(t))) = \int_{\Sigma^t_{V_0}} H\, dA$ and $\frac{dV}{dt} = A_{V_0}(V(t))$, so that

$$\frac{d}{dV}(A_{V_0}(V)) = A'_{V_0}(V) = \frac{\int_{\Sigma^t_{V_0}} H\, dA}{A_{V_0}(V(t))}.$$

By the second variation of area formula we obtain (since $\eta = 1$)

$$A_{V_0}(V_0)^2 A''_{V_0}(V_0) = \int_{\Sigma(V_0)} (-\|I\!I\|^2 - Ric(\nu,\nu))dA.$$

Taking the trace of the Gauss equation gives $Ric(\nu,\nu) = \frac{1}{2}R - K + \frac{1}{2}(H^2 - \|I\!I\|^2)$, where $R = R(g)$ is the scalar curvature of the ambient three-space and K is the Gauss curvature of the surface. We obtain

$$A_{V_0}(V_0)^2 A''_{V_0}(V_0) = \int_{\Sigma(V_0)} (-\tfrac{1}{2}R + K - \tfrac{1}{2}H^2 - \tfrac{1}{2}\|I\!I\|^2)dA.$$

Since $\Sigma(V_0)$ has only one component by assumption, $\int_{\Sigma(V_0)} K\, dA = 2\pi\chi(\Sigma(V_0)) \leq 4\pi$ by the Gauss-Bonnet theorem. Since $\|I\!I\|^2 \geq \frac{1}{2}H^2$, we arrive at the inequality

$$A_{V_0}(V_0)^2 A''_{V_0}(V_0) \leq 4\pi + \int_{\Sigma(V_0)} (-\tfrac{1}{2}R - \tfrac{3}{4}H^2)\, dA. \tag{6.1}$$

We will apply this inequality below.

6.2. Penrose inequality for some solutions of the Einstein-Maxwell constraints.

We now discuss the Penrose Inequality in the context of a certain class of solutions to the Einstein-Maxwell constraints. As noted in [WY], in the case of a connected horizon, the Huisken-Ilmanen proof can be carried through to prove the Penrose inequality that we discuss below, under less restrictive assumptions. We also remark that recently Weinstein and Yamada [WY] showed that for multiple-component horizons, a natural related Penrose inequality fails.

Proposition 6.1. *Assume (M, g, E) is an asymptotically flat solution of the Einstein-Maxwell constraints $R(g) = 2|E|^2$, $div_g(E) = 0$, which outside a compact set agrees with Reissner-Nordstrom data on the exterior of a ball, with mass m and charge Q, and $m > |Q|$. Suppose furthermore that M has only one horizon Σ_0 and admits a connected isoperimetric profile (with respect to Σ_0) $\Sigma(V)$, so that for sufficiently large V, $\Sigma(V)$ is a spherically symmetric sphere in Reissner-Nordstrom. Then $m \geq \sqrt{\frac{A(\Sigma_0)}{16\pi}} + \frac{Q^2}{2r_0} = \frac{1}{2}(r_0 + \frac{Q^2}{r_0})$, where r_0 is defined by $A(\Sigma_0) = 4\pi r_0^2$.*

Proof. We have established the isoperimetric profile for Reissner-Nordstrom in Cor. 3.3. We discuss the calculations that relate the mass to the Hawking mass of the isoperimetric surfaces for the model. Since solutions (g, E) of the Einstein-Maxwell constraints have nonnegative scalar curvature $R(g) = 2|E|^2$, we have from Eq. (6.1), $A_{V_0}(V_0)^2 A''_{V_0}(V_0) \leq 4\pi - \int_{\Sigma(V_0)}(|E|^2 + \frac{3}{4}H^2)\, dA = 4\pi - \frac{3}{4}H^2 A_{V_0}(V_0) - \int_{\Sigma(V_0)} |E|^2\, dA$. Since E is divergence-free, the flux integral $\int_\Sigma E^i \nu_i\, dA$ is a homological invariant, and thus is just $4\pi Q$. The preceding inequality thus yields (using Cauchy-Schwarz)

$$A_{V_0}(V_0)^2 A''_{V_0}(V_0) \leq 4\pi - \tfrac{3}{4}H^2 A_{V_0}(V_0) - \frac{(4\pi Q)^2}{A(V_0)}.$$

Since $A'_{V_0}(V_0) = H$, then as noted above this can be interpreted as a weak formulation of

$$A''(V) \leq \frac{4\pi}{A(V)^2} - \frac{3A'(V)^2}{4A(V)} - \frac{(4\pi Q)^2}{A(V)^3}.$$

Equivalently, for $F = A^{3/2}$ we have

$$F''(V) \leq \frac{36\pi - F'(V)^2 - 144\pi^2 Q^2 F(V)^{-2/3}}{6F(V)}. \tag{6.2}$$

We will work with the mass function $m(V)$, defined by

$$m(V) = \frac{F(V)^{1/3}}{144\pi^{3/2}}\left(36\pi - F'(V)^2\right) + \sqrt{\pi}\, Q^2 F(V)^{-1/3}.$$

If $F(V)$ were smooth, we would have

$$m'(V) = \frac{\frac{1}{3}F'(V)(F(V))^{-2/3}}{144\pi^{3/2}}\big((36\pi - (F'(V))^2) - 6F(V)F''(V) - \sqrt{\pi}Q^2(F(V))^{-2/3}\big).$$

Since (6.2) holds, and since $F(V)$ is nondecreasing (since there is only one horizon), $m(V)$ is an non-decreasing function; actually this statement requires some care to prove, since the function $F(V)$ may fail to be smooth, so one would need to check directly that $m'(V) \geq 0$ in the sense of distributions, *i.e* by pairing with appropriate test functions. We omit the details.

If $Q = 0$, the mass function is the Hawking mass of the isoperimetric surface bounding a volume V, since $F'(V) = \frac{3}{2}A(V)^{1/2}A'(V) = \frac{3}{2}A(V)^{1/2}H$ implies

$$\begin{aligned} m(V) &= \frac{A(V)^{1/2}}{144\pi^{3/2}}\left(36\pi - \tfrac{9}{4}A(V)H^2\right) + \sqrt{\pi}\, Q^2 F(V)^{-1/3} \\ &= \sqrt{\frac{A(V)}{16\pi}}\left(1 - \int_{\Sigma(V)} \frac{H^2}{16\pi}\right) + \sqrt{\pi}\, Q^2 F(V)^{-1/3}. \end{aligned}$$

Since $H = 0$ at the horizon, $m(0) = \sqrt{\frac{A(0)}{16\pi}} + \frac{Q^2}{2r_0}$. For V sufficiently large $\Sigma(V)$ is a radial sphere S_r in Reissner-Nordstrom, so that $m(V)$ is the Hawking mass of S_r plus the charge term:

$$m(V) = \sqrt{\frac{4\pi r^2}{16\pi}}\left(1 - 4\pi r^2 \cdot \frac{1 - \frac{2m}{r} + \frac{Q^2}{r^2}}{4\pi r^2}\right) + \frac{Q^2}{2r} = m.$$

Hence $m = \lim\limits_{V\to+\infty} m(V) \geq m(0)$, giving us a Penrose Inequality with charge: $m \geq \sqrt{\frac{A(0)}{16\pi}} + \frac{Q^2}{2r_0} = \frac{1}{2}(r_0 + \frac{Q^2}{r_0})$. $\square$

We now briefly sketch how to use this result to conclude the Penrose inequality holds for more general asymptotically flat solutions (M, g, E) of the Einstein-Maxwell constraints. We cite a condition (C1) from [Br1]: there is only one horizon, and for $V > 0$, if one or more isoperimetric surfaces exists for this volume V, then at least one of these surfaces has only one component. This condition is not required in [Br2], [HI], for which if there is more than one horizon, one considers the *outermost* horizons in any end. We have an approximation result from [C] which allows us to normalize the asymptotics: asymptotically flat solutions (M, g, E) of the Einstein-Maxwell constraints admit approximations by data which agree with suitable Reissner-Nordstrom data in each end, where the perturbation is localized near infinity. Assuming condition (C1) holds after this perturbation, one shows that the isoperimetric surfaces $\Sigma(V)$ exist and agree with those of Reissner-Nordstrom for sufficiently large V. The proof of these claims should actually follow from the proofs in [Br1] for the Schwarzschild case; much of the construction relies on the geometry being asymptotically flat and spherically symmetric near infinity, and a main technical theorem which is used in the proof is an inequality in Euclidean space, which carries over to Schwarzschild (as used in [Br1]) and Reissner-Nordstrom for large radii by perturbation. Since the Penrose inequality $m \geq \sqrt{\frac{A(0)}{16\pi}} + \frac{Q^2}{2r_0}$ in this case also follows from [HI], we omit the technical details.

6.3. **On the Penrose inequality for asymptotically Schwarzschild AdS spaces.** In this section we show that the analogous mass function $m(V)$ (if it exists) will be nondecreasing in an asymptotically Schwarzschild AdS space. We remark that in general the mass of asymptotically hyperbolic spaces is more subtle than for asymptotically flat spaces; *cf.* [CH], [W], [Z]. We are only discussing below a class of asymptotically hyperbolic spaces with a spherical infinity and with special asymptotics.

We have the following proposition; by *horizon* we mean here that Σ_0 has (inward) mean curvature $H = 2$ [BC].

Proposition 6.2. *Assume (M, g) is a three-manifold with $R(g) \geq -6$, which outside a compact set is isometric to an exterior of a ball in Schwarzschild AdS space of mass $m > 0$. Suppose furthermore that M has only one horizon Σ_0 and admits a connected isoperimetric profile (with respect to Σ_0) $\Sigma(V)$, so that for sufficiently large V, $\Sigma(V)$ is the spherically symmetric sphere in Schwarzschild AdS of volume V. Then $m \geq \sqrt{\frac{A(\Sigma_0)}{16\pi}}$.*

Proof. Schwarzschild AdS is asymptotic to hyperbolic three-space, so the definitions and computations change slightly from above. We begin by putting $R(g) \geq -6$ into the inequality (6.1) to obtain

$$\begin{aligned} A_{V_0}(V_0)^2 A''_{V_0}(V_0) &\leq 3A_{V_0}(V_0) + 4\pi - \int_{\Sigma(V_0)} \tfrac{3}{4} H^2 \, dA \\ &= 3A_{V_0}(V_0) + 4\pi - \tfrac{3}{4} H^2 A_{V_0}(V_0). \end{aligned}$$

Hence

$$A''_{V_0}(V_0) \leq \frac{3}{A_{V_0}(V_0)} + \frac{4\pi}{A_{V_0}(V_0)^2} - \frac{3A'_{V_0}(V_0)^2}{4A_{V_0}(V_0)}.$$

Since by definition $A(V_0) = A_{V_0}(V_0)$ and $A(V) \leq A_{V_0}(V)$, we have the weak inequality

$$A''(V) \leq \frac{3}{A(V)} + \frac{4\pi}{A(V)^2} - \frac{3A'(V)^2}{4A(V)},$$

or equivalently, for $F = A^{3/2}$

$$F''(V) \leq \frac{27F(V)^{2/3} + 36\pi - F'(V)^2}{6F(V)}. \tag{6.3}$$

We modify the Hawking mass in this setting with one extra term which accounts for the non-minimal horizon, and so we get a corresponding $m(V)$ for the isoperimetric surfaces as follows:

$$\begin{aligned} m(V) &= \sqrt{\frac{A(V)}{16\pi}}\left(1+\frac{A(V)}{4\pi}-\int_{\Sigma(V)}\frac{H^2}{16\pi}\,dA\right) \\ &= \frac{A(V)^{1/2}}{16\pi^{3/2}}\left(4\pi+A(V)-\tfrac{1}{4}A(V)(A'(V))^2\right) \\ &= \frac{F(V)^{1/3}}{16\pi^{3/2}}\left(4\pi+F(V)^{2/3}-\tfrac{1}{4}F(V)^{2/3}(\tfrac{2}{3}F(V)^{-1/3}F'(V))^2\right) \\ &= \frac{F(V)^{1/3}}{144\pi^{3/2}}\left(36\pi+9F(V)^{2/3}-F'(V)^2\right). \end{aligned}$$

The reason for the modification is that since the (inward) mean curvature of Σ_0 is 2, we again have $m(0)=\sqrt{\frac{A(0)}{16\pi}}$.

Since (6.3) holds (and again, $F(V)$ is nondecreasing since there is only one horizon), we have the distributional inequality

$$m'(V)=\tfrac{2}{144\pi^{3/2}}F(V)^{1/3}F'(V)\left(-F''(V)+\frac{36\pi+27F(V)^{2/3}-F'(V)^2}{6F(V)}\right)\geq 0.$$

For V sufficiently large, $m(V)$ is the Hawking mass of some radially symmetric sphere $S_r=\Sigma(V)$ and thus

$$m(V)=\sqrt{\frac{4\pi r^2}{16\pi}}\left(1+\frac{4\pi r^2}{4\pi}-\frac{4\pi r^2}{16\pi}\cdot\frac{4(1+r^2-\frac{2m}{r})}{r^2}\right)=m.$$

Hence $m=\lim_{V\to+\infty}m(V)\geq m(0)=\sqrt{\frac{A(0)}{16\pi}}$, giving us the desired Penrose Inequality. $\square$

7. Conclusions

We conjecture that there exists a reasonable class of spaces with $R(g)\geq -6$ which are asymptotically Schwarzschild AdS for which the above analysis will yield a Penrose Inequality. We hope to report on this in a future work. Although the class would be limited in several respects, it is interesting problem, following the work of Bray and in light of the recent work of Huisken [H1], [H2], to understand better the relationship of the mass to the isoperimetric properties of the space.

We mention that foliations near infinity of constant mean curvature (CMC) have appeared in the context of relativity [HY], [M], [QT], [Y]. It is tempting to conjecture that these uniquely-determined foliations near infinity by constant mean curvature spheres give the isoperimetric profiles.

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9. Appendix: Metric Formulas

Consider a metric of the form $g = f(r)dr^2 + r^2 d\Omega^2 = f(r)dr^2 + r^2 d\phi^2 + r^2\sin^2(\phi)d\theta^2$ with $f(r) > 0$. We collect here the basic geometric formulas which we apply to our three families of metrics above. We use the Einstein summation convention below, and the indices $(1,2,3)$ correspond to the variables (r,ϕ,θ).

9.1. **Christoffel symbols.** We display the metric and its inverse in matrix form:

$$(g_{ij}) = \begin{pmatrix} f(r) & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\phi) \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} \frac{1}{f(r)} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2(\phi)} \end{pmatrix}.$$

We recall the formula $\Gamma^k_{ij} = \frac{1}{2}g^{mk}(g_{jm,i} + g_{mi,j} - g_{ij,m})$ for the Christoffel symbols. We can simplify our calculations by making two observations. Since g_{ij} and g^{ij} are diagonal, we have $\Gamma^k_{ij} = \frac{1}{2}g^{kk}(g_{jk,i} + g_{ki,j} - g_{ij,k})$, and $\Gamma^k_{ij} = 0$ when i, j, and k are all distinct. For reference here are the nonzero Christoffel symbols:

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2}g^{11}(g_{11,1} + g_{11,1} - g_{11,1}) = \frac{f'(r)}{2f(r)}, \\
\Gamma^1_{22} &= \frac{1}{2}g^{11}(g_{21,2} + g_{12,2} - g_{22,1}) = -\frac{r}{f(r)}, \\
\Gamma^1_{33} &= \frac{1}{2}g^{11}(g_{31,3} + g_{13,3} - g_{33,1}) = -\frac{r\sin^2(\phi)}{f(r)}, \\
\Gamma^2_{33} &= \frac{1}{2}g^{22}(g_{32,3} + g_{23,3} - g_{33,2}) = -\sin(\phi)\cos(\phi), \\
\Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{2}g^{22}(g_{22,1} + g_{21,2} - g_{12,2}) = \frac{1}{r}, \\
\Gamma^3_{13} = \Gamma^3_{31} &= \frac{1}{2}g^{33}(g_{33,1} + g_{31,3} - g_{13,3}) = \frac{1}{r}, \\
\Gamma^3_{23} = \Gamma^3_{32} &= \frac{1}{2}g^{33}(g_{33,2} + g_{32,3} - g_{23,3}) = \cot(\phi).
\end{aligned}$$

9.2. **Second fundamental form and mean curvature of radial spheres S_r.** We compute the second fundamental form and mean curvature of the coordinate spheres of constant r. Let Z^N be the normal projection of a vector Z. We have

$$\begin{aligned}
B(\partial_\phi,\partial_\phi) &= (\nabla_{\partial_\phi}\partial_\phi)^N = (\Gamma^1_{22}\partial_r + \Gamma^2_{22}\partial_\phi + \Gamma^3_{22}\partial_\theta)^N = -\tfrac{r}{f(r)}\partial_r, \\
B(\partial_\phi,\partial_\theta) &= (\nabla_{\partial_\phi}\partial_\theta)^N = (\Gamma^1_{23}\partial_r + \Gamma^2_{23}\partial_\phi + \Gamma^3_{23}\partial_\theta)^N = 0, \\
B(\partial_\theta,\partial_\theta) &= (\nabla_{\partial_\theta}\partial_\theta)^N = (\Gamma^1_{33}\partial_r + \Gamma^2_{33}\partial_\phi + \Gamma^3_{33}\partial_\theta)^N = -\tfrac{r\sin^2\phi}{f(r)}\partial_r.
\end{aligned}$$

Let $N = -\nu$ denote the inward unit normal vector field to S_r. Then $g(\partial_r, N) = -\|\partial_r\| = -\sqrt{f(r)}$, so the second fundamental form $I\!I$, defined by $I\!I(V,W) = g(B(V,W),N)$, is given by

$$I\!I(\partial_\phi,\partial_\phi) = \frac{r}{\sqrt{f(r)}}, \qquad I\!I(\partial_\phi,\partial_\theta) = 0, \qquad I\!I(\partial_\theta,\partial_\theta) = \frac{r\sin^2(\phi)}{\sqrt{f(r)}}.$$

Thus the mean curvature for S_r, which is constant by symmetry, is

$$H_{S_r} = g^{\phi\phi} I\!I(\partial_\phi, \partial_\phi) + g^{\theta\theta} I\!I(\partial_\theta, \partial_\theta) = \left(\frac{\frac{r}{\sqrt{f(r)}}}{r^2} + \frac{\frac{r\sin^2\phi}{\sqrt{f(r)}}}{r^2 \sin^2\phi} \right) = \frac{2}{r\sqrt{f(r)}}.$$

9.3. **Ricci and scalar curvature.** We use the formulas $R_{ij} = R^l_{ilj}$ and $R^l_{ikj} = \Gamma^l_{ij,k} - \Gamma^l_{ik,j} + \Gamma^m_{ij}\Gamma^l_{km} - \Gamma^m_{ik}\Gamma^l_{jm}$. A simple computation shows that the Ricci tensor is diagonal in this coordinate system, and the diagonal entries are given as follows:

$$\begin{aligned}
R_{11} &= R^1_{111} + R^2_{121} + R^3_{131} = 0 + \tfrac{f'(r)}{2rf(r)} + \tfrac{f'(r)}{2rf(r)}, \\
R_{22} &= R^1_{212} + R^2_{222} + R^3_{232} = \tfrac{rf'(r)}{2[f(r)]^2} + 0 + \left(1 - \tfrac{1}{f(r)}\right), \\
R_{33} &= R^1_{313} + R^2_{323} + R^3_{333} = \sin^2(\phi)\left[\tfrac{rf'(r)}{2[f(r)]^2} + \left(1 - \tfrac{1}{f(r)}\right) + 0\right].
\end{aligned}$$

Thus we find the scalar curvature is

$$R(g) = g^{ij}R_{ij} = g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} = \frac{2f'(r)}{r[f(r)]^2} + \frac{2}{r^2} - \frac{2}{r^2 f(r)}.$$

References

[ABR] Axler, S., Bourdon, P., Ramey, W.: *Harmonic Function Theory.* New York: Springer-Verlag, 1992

[Br1] Bray, H.L.: *The Penrose Inequality in General Relativity and Volume Comparison Theorems Involving Scalar Curvature.* Thesis. Stanford University: 1997

[Br2] Bray, H.L.: Proof of the Riemannian Penrose Inequality Using the Positive Mass Theorem. J. Diff. Geom. **59**, no. 2, 177-267 (2001)

[Br3] Bray, H.L.: Black holes, geometric flows, and the Penrose inequality in general relativity. Notices Amer. Math. Soc. **49**, no. 11, 1372-1381 (2002)

[BC] Bray, H.L., Chruściel, P.T.: The Penrose Inequality. In: *The Einstein equations and the large scale behavior of gravitational fields.* pp. 39-70. Basel: Birkhäuser, 2004. ArXiv: gr-qc/0312047 (2003)

[BM] Bray, H.L., Morgan, F.: An isoperimetric comparison theorem for Schwarzschild space and other manifolds. Proc. AMS **130**, no.5, 1467-1472 (2002)

[Ch] Chavel, I.: *Riemannian Geometry: A Modern Introduction.* New York: Cambridge University Press, 1993

[CH] Chruściel, P.T., Herzlich, M.: The mass of asymptotically hyperbolic Riemannian manifolds. Pacific J. Math. **212**, no. 2, 231-264 (2003)

[C] Corvino, J.: On the asymptotics of the Einstein-Maxwell constraint equations. *In preparation*

[G] Geroch, R.: Energy Extraction. Ann. New York Acad. Sci. **224** 108-117 (1973)

[HHM] Howards, H., Hutchings, M., Morgan, F.: The isoperimetric problem on surfaces. Amer. Math. Monthly **106**, 430-439 (1999)

[H1] Huisken, G.: Radial foliations of asymptotically flat 3-manifolds. Lecture, Einstein Constraint Equations Conference, Isaac Newton Institute, 2005. http://www.newton.cam.ac.uk/webseminars/pg+ws/2005/gmr/gmrw03/1212/huisken/

[H2] Huisken, G: An isoperimetric concept for mass and quasilocal mass. Oberwolfach Rep. **3**, no.1, 87-88 (2006)

[HI] Huisken, G., Ilmanen, T.: The inverse mean curvature flow and the Riemannian Penrose Inequality. J. Diff. Geom. **59**, no. 3, 353-437 (2001)

[HY] Huisken, G., Yau, S.-T.: Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature. Invent. Math. **124** 281-311 (1996)

[M] Metzger, J.: Foliations of asymptotically flat three-manifolds with two-surfaces of prescribed mean curvature. *Preprint.* ArXiv:math.DG/0410413

[QT] Qing, J., Tian, G.: On the uniqueness of the foliation of spheres of constant mean curvature in asymptotically flat three-manifolds. *Preprint.* ArXiv:math.DG/0506005

[T] Taylor, M.: Partial Differential Equations III. New York: Springer-Verlag, 1996

[W] Wald, R. M.: *General Relativity.* Chicago: The University of Chicago Press, 1984

[W] Wang, X.: The mass of asymptotically hyperbolic manifolds. Jour. Diff. Geom. **57** no. 2, 273-299 (2001)

[WY] Weinstein, G., Yamada, S.: On a Penrose inequality with charge. Comm Math. Phys. **257** 703-723 (2005)

[Y] Ye, R.: Foliation by constant mean curvature spheres on asymptotically flat manifolds. In: *Geometric Analysis and the Calculus of Variations*. pp. 369-383. Cambridge, MA: International Press, 1996. ArXiv: dg-ga/9709020

[Z] Zhang, X.: A definition of total energy-momenta and the positive mass theorem on asymptotically hyperbolic three-manifolds. I. Comm. Math. Phys. **249**, no. 3, 529-548 (2004)

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