ON THE CENTER OF MASS OF ISOLATED SYSTEMS

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Abstract. We discuss the center of mass of asymptotically flat manifolds. Our main result is that for a class of metrics that includes those which near infinity are conformally flat with vanishing scalar curvature and positive mass, the Huisken-Yau geometric center of mass agrees with the center of mass defined by the ADM formulation of the initial value problem for Einstein's equation.

1. Introduction

To model isolated gravitational systems, one often uses asymptotically flat metrics as the initial data in the Cauchy problem for Einstein's equation. A quantity called the *mass* (or the *energy*) can be attributed to such metrics [ADM], and is an invariant of the asymptotically flat structure [Ba1]. In fact, such data possess a well-defined energy-momentum four-vector, and, in the presence of approximate parity conditions, *cf.* [BO], [RT], the data can also be assigned a center of mass and an angular momentum. The mass has geometric significance, and the importance of this quantity and its ties to geometry are well known, *e.g.* [SY1], [LP]. A natural question is whether the center of mass can be described in a geometrically interesting and intrinsic way.

This problem has been addressed from two different approaches, both of which yield a geometric notion of a center in terms of a unique foliation near infinity by constant mean curvature spheres. Ye [Y] established such a foliation using a geometric singular perturbation (implicit function theorem) argument. Huisken and Yau [HY] used mean curvature flow to show that asymptotically flat manifolds with positive mass have an intrinsically defined center of mass, in terms of a unique foliation of a neighborhood of infinity by *stable* constant mean curvature spheres. We note recent work by Qing and Tian on the uniqueness of the foliation [QT], work of Rigger [R] and Neves and Tian [NT1], [NT2] on the asymptotically hyperbolic case, and work of Metzger [M] on prescribed mean curvature foliations. A natural problem is to relate this geometric notion of the center with the center of mass from the ADM formulation of the initial value problem in general relativity, which was used, for example, in the asymptotic gluing constructions of Corvino-Schoen [C1], [CS]. We establish agreement of these notions in the case where the metric near infinity is conformally flat and has vanishing scalar curvature, which answers a question raised by Zhang [Z] (among others). This class of metrics is dense in the space of asymptotically flat metrics of nonnegative scalar curvature [SY2], a space of interest for physics and for geometry.

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Indeed, the initial geometric data for the Einstein equation is a triple (M, g, K), where M is an oriented three-manifold, g is a Riemannian metric on M, and K is a symmetric two-tensor; there may also be initial data for the matter fields. The Gauss and Codazzi equations yield compatibility conditions for g and K to be the induced metric and second fundamental forms of M embedded in a four-dimensional Lorentzian space $(\mathcal{S}, \bar{g})$ satisfying the Einstein equation $Ric(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} = 8\pi T$. We note that in our units $G = 1$ and $c = 1$, and we have taken vanishing cosmological constant. The vacuum case $(T = 0)$ reduces to $Ric(\bar{g}) = 0$. The Gauss equation yields the Hamiltonian constraint $R(g) - |K|^2 + H^2 = 16\pi\mu$, where μ is the local energy density from the matter fields, $R(g)$ is the scalar curvature of the slice (M, g), $|K|^2 = g^{ij}g^{kl}K_{ik}K_{jl}$ and $H = g^{ij}K_{ij}$ (we use the Einstein summation convention). Thus we see in the maximal case $H = 0$, we have $R(g) = 16\pi\mu + |K|^2$; in this case, nonnegative local energy density implies nonnegative scalar curvature. In the time-symmetric $(K = 0)$ vacuum case, the constraint reduces to $R(g) = 0$.

One often models isolated systems by using asymptotically flat data (M, g, K), where M has a compact subset C so that $M \setminus C$ is the disjoint union of a finite number of exterior regions, each diffeomorphic to the exterior of a ball in $\mathbb{R}^3$, with each admitting coordinates near infinity in which the metric components g_{ij} decay to the Euclidean δ_{ij} and K decays to zero in an appropriate fashion. An energy quantity often called the *mass* was identified in the study of the Hamiltonian formulation of general relativity [ADM] (*cf.* [Ba2]) and is given by

$$m = \frac{1}{16\pi} \lim_{R\to\infty} \int_{|x|=R} \sum_i \left(g_{ij,i} - g_{ii,j}\right) \nu_e^j d\mu_e \tag{1.1}$$

where $d\mu_e$ is Euclidean surface measure, ν_e is the outward Euclidean normal, and the commas denote partial derivatives. Note that the Einstein convention is in force, so we are indeed summing over both i and j above. This quantity is an invariant of the asymptotically flat structure of the three-manifold ([Ba1]), and is actually a component of a Lorentz-invariant energy-momentum four-vector. Moreover, the mass is nonnegative for asymptoticaly flat spacetimes satisfying the dominant energy condition [SY1]; in fact, the mass is positive for (M, g) asymptotically flat with nonnegative scalar curvature, unless (M, g) is isometric to Euclidean space $(\mathbb{R}^3, g_e)$, in which case the mass vanishes. In the course of establishing the Positive Mass Theorem, Schoen and Yau [SY2] proved that among asymptotically flat three-metrics of nonnegative scalar curvature, metrics which near infinity are conformally flat and have vanishing scalar curvature are dense in a topology for which the mass function is continuous; an analogous result for the full energy-momentum vector was established in [CS]. Such metrics admit a coordinate chart near infinity in which the metric has the form $g_{ij} = (1 + \frac{m}{2|x|})^4\delta_{ij} + O(|x|^{-2})$. The leading term is the well-known Schwarzschild metric g^S, and the quantity m agrees with that in Equation (1.1). We thus consider classes of metrics which in appropriate asymptotically flat coordinates near infinity can be written $g_{ij}(x) = (1+\frac{m}{2|x|})^4\delta_{ij} + O_k(|x|^{-2})$, where $f \in O_k(|x|^{-2})$ provided $\partial^\alpha f \in O(|x|^{-2-|\alpha|})$, for $|\alpha| \leq k$. Thus the α^{th} partial derivative will agree with the Schwarzschild derivative up to error terms which are $O(|x|^{-2-|\alpha|})$.

It is important to emphasize that this mass quantity includes the energy due to the gravitational field itself, which is not given by any local density. In particular, the energy integral in Equation (1.1) is positive even in the vacuum case, unless

(M, g) is flat Euclidean geometry $(\mathbb{R}^3, g_e)$. In the next section we will discuss a similar formulation for the center of mass.

Before we proceed, we introduce some notation and conventions. For any Riemannian manifold (M, g), we have the induced connection ∇, the volume measure dv_g induced by g, and the induced measure $d\mu_g$ on hypersurfaces; we let dv_e and $d\mu_e$ be the respective measures for the Euclidean case $g = g_e$. We recall the standard function space norms

$$\begin{aligned} \|u\|_{C^0} &= \sup_M |u| \\ \|u\|_{L^p(M)} &= \left(\int_M |u|^p \, dv_g \right)^{1/p} \\ \|u\|_{W^{1,p}(M)} &= \|u\|_{L^p(M)} + \|du\|_{L^p(M)}. \end{aligned}$$

We abbreviate coordinate vectors $\frac{\partial}{\partial x^j} = \partial_j$. We follow the standard convention of using a semicolon for covariant derivatives, and (as noted above) using a comma for partial derivatives. We take the Laplacian to be $\Delta f = g^{ij} f_{;ij}$; when there are multiple metrics under consideration, we may use subscripts on the connection and Laplacian (∇_g, Δ_g), for clarity. Our convention on the curvature tensor $R(\partial_i, \partial_j, \partial_k) = R^l_{ijk} \partial_l$ is such that the Ricci identity is

$$Z_{i;kj} - Z_{i;jk} = R^l_{jki} Z_l.$$

If R^Σ is the Riemann tensor for a hypersurface Σ in (M, g), then the Gauss equation is given in a local orthonormal frame $\{e_1, e_2\}$ on Σ by

$$\langle R(e_1, e_2, e_1), e_2 \rangle = \langle R^\Sigma(e_1, e_2, e_1), e_2 \rangle + h_{12}^2 - h_{11} h_{22},$$

where h_{ij} are the components of the second fundamental form of Σ, and where the metric g is denoted with the bracket pairing. Finally in any coordinate chart, we let $r = |x|$ be the Euclidean length of the coordinate vector x.

2. The center of mass for solutions of the Hamiltonian constraint

In this section we discuss a flux integral for the center of mass akin to the mass integral, for solutions to the Hamiltonian constraint. We identify this center in the spherical harmonic expansion of the conformal factor for metrics which are conformally Euclidean with vanishing scalar curvature near infinity, and identify the mass and center via conformal symmetries near infinity. We begin with some motivation, by comparison to the Newtonian case. We also note that in this section, Δ is the Euclidean Laplacian.

For metrics g which are conformally flat near infinity, say $g_{ij} = u^4 \delta_{ij}$ outside a compact set, the scalar curvature $R(g) = -8u^{-5} \Delta u$ vanishes in this region if and only if u is harmonic. For u tending to 1 near infinity, we can expand the conformal factor using spherical harmonics: $u(x) = 1 + \frac{A}{|x|} + O_\infty(|x|^{-2})$. It is elementary to compute

$$\int\limits_{|x|=R} \sum_i (g_{ij,i} - g_{ii,j}) \nu_e^j \, d\mu_e = -8 \int\limits_{|x|=R} \frac{\partial u}{\partial r} \, d\mu_e + O(1/R) = 16\pi m + O(1/R),$$

where $m = 2A$ is thus seen to be the ADM mass.

In the Newtonian setting, we can study gravitational systems in terms of the mass density ρ, with decay conditions on ρ corresponding to the system being isolated; for example, we might take ρ to be compactly supported. This gives rise to a gravitational potential ϕ which satisfies $\Delta\phi = 4\pi\rho$ (recall that we have taken $G=1$). In case ρ is compactly supported, ϕ is harmonic near infinity, and thus it can be expanded as a series in spherical harmonics, which yields (we take ϕ to decay to zero at infinity) $\phi(x) = -\frac{m}{|x|} + O_\infty(|x|^{-2})$. Note that the total mass of the system can be written as a flux integral in terms of the potential:

$$\begin{aligned}\int_{\mathbb{R}^3} \rho \, dv_e &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \Delta\phi \, dv_e \\ &= \frac{1}{4\pi}\lim_{R\to+\infty}\int_{|x|=R} \frac{\partial\phi}{\partial r}\, d\mu_e = m.\end{aligned}$$

Compare this to the situation of time-symmetric vacuum data given by $g = u^4\gamma$ for a perturbation $\gamma = g_e + h$ of the Minkowski data by sufficiently small and compactly supported h, with u tending to 1 at infinity. The Hamiltonian constraint $R(g)=0$ in this case becomes

$$\Delta_\gamma u = \tfrac{1}{8}R(\gamma)u.$$

Near infinity u is harmonic, so that $u = 1 + \frac{A}{|x|} + O_\infty(|x|^{-2})$, and we have as above

$$\begin{aligned} m &= -\frac{1}{2\pi}\lim_{R\to+\infty}\int_{|x|=R}\frac{\partial u}{\partial r}\, d\mu_e = 2A \\ &= -\frac{1}{2\pi}\int\limits_{\mathbb{R}^3} \Delta_\gamma u \, dv_\gamma \\ &= -\frac{1}{16\pi}\int_{\mathbb{R}^3} R(\gamma) u \, dv_\gamma.\end{aligned}$$

The Positive Mass Theorem implies m is positive, unless g is isometric to the Euclidean metric; this can happen if the deformation comes from a diffeomorphism, for example. Deformation tensors with nonzero transverse-traceless components in the York decomposition generate *vacuum* deformations of the flat metric with positive mass (*cf.* [BD], [CM], [C2]).

We now turn to the center of mass. In Newtonian gravity, the center of mass c in a coordinate system is given by

$$mc^k = \int_{\mathbb{R}^3} x^k \rho \, dv_e.$$

For compactly supported ρ, we again expand the potential $\phi(x) = -\frac{m}{|x|} - \frac{\Gamma_i x^i}{|x|^3} + O_\infty(|x|^{-3})$ and find the center in terms of a flux integral of ϕ, using Green's identity:

$$\begin{aligned} 4\pi m c^k &= \int_{\mathbb{R}^3} x^k \Delta\phi \, dv_e \\ &= \lim_{R\to+\infty}\int_{|x|=R}\left(x^k\frac{\partial\phi}{\partial r} - \frac{x^k}{R}\,\phi\right) d\mu_e \\ &= 4\pi\Gamma_k.\end{aligned}$$

Again we see that the relevant property of the distribution ρ manifests itself in the potential ϕ. We will see a similar situation holds in the case of general relativity.

Indeed for metrics $g_{ij} = (1+\frac{m}{2|x|})^4\delta_{ij} + O_2(|x|^{-2})$ *which solve the Hamiltonian constraint* $R(g) - |K|^2 + H^2 = 0$, and with $K = O(|x|^{-2})$ (and with K parity-asymmetric to $O(|x|^{-3})$, *i.e.* $K(x)+K(-x) = O(|x|^{-3})$), a center of mass quantity related in general to the angular momentum of the spacetime is given by the following:

$$\frac{1}{16\pi}\lim_{R\to\infty}\int\limits_{|x|=R}\left[\sum_i x^k\left(g_{ij,i}-g_{ii,j}\right)\nu_e^j d\mu_e-\sum_i\left(g_{ik}\nu_e^i-g_{ii}\nu_e^k\right)d\mu_e\right]. \tag{2.1}$$

In terms of the (skew-symmetric) angular momentum tensor $M_{\mu\nu}$, the integral above is (up to a constant factor) M_{0k}, *cf.* [RT], [BO], [CD]. That this limits exists is shown by integrating the Hamiltonian constraint against the function x^i over the annulus $\{R\le|x|\le 2R\}$, expanding the scalar curvature as $R(g)=\sum_{i,j}(g_{ij,ij}-g_{ii,jj})+O(|x|^{-4})$, and using the approximate parity symmetry of the data (g,K). The center of mass integral is just the boundary term obtained by integration by parts ([RT], [CS]). We remark that the parity symmetry required to give a finite center of mass integral has its analogue in Newtonian gravity, where it is easy to find a density function ρ decaying to zero at infinity so that the total mass is finite, but so that one or more moments are not, *e.g.* consider any smooth positive ρ which near infinity is of the form $\frac{C}{r^{3.5}}+\frac{x}{r^5}$. We note that one can allow the metric to have a more general form than above, with approximate parity symmetry imposed to ensure finite angular momentum, (*cf.* [BO], [RT], [CS]). For such data (g,K), it follows from [CS] that given *any* compact subset, the data can be approximated by data (solving the constraints) which near (spatial) infinity is precisely the initial data for a slice in a Kerr space-time, and for which the energy, linear momentum, angular momentum and center of mass are arbitrarily close to that of the given data, and for which the two data sets agree inside the given compact set (*cf.* [CD]).

Furthermore, note that if we translate our coordinate system by adding a constant vector a^k to the coordinate functions, the chart is still asymptotically flat, and the limit of the center of mass integral above changes by an additional term

$$\frac{1}{16\pi}\lim_{R\to+\infty}\int\limits_{|x|=R}\sum_i a^k\left(g_{ij,i}-g_{ii,j}\right)\nu_e^j d\mu_e=ma^k. \tag{2.2}$$

The center of mass can be determined from the expansion of the conformal factor in the case when the metric g is conformally flat and scalar flat near infinity: outside a compact set, $g_{ij}=u^4\delta_{ij}$ with u harmonic and tending to 1 near infinity. In the conformally flat case, the center of mass integrand simplifies considerably:

$$\begin{aligned}\sum_i x^k\left(g_{ij,i}-g_{ii,j}\right)\nu_e^j-\sum_i\left(g_{ik}\nu_e^i-g_{ii}\nu_e^k\right) &= x^k(-8u^3u_{,j}\nu_e^j)+2u^4\nu_e^k\\ &= -8x^ku^3\frac{\partial u}{\partial r}+\frac{2u^4x^k}{|x|}.\end{aligned}$$

If we again expand the conformal factor using spherical harmonics as $u(x)=1+\frac{A}{|x|}+\frac{B_ix^i}{|x|^3}+O_\infty(|x|^{-3})$, we see $u^3=1+\frac{3A}{|x|}+O(|x|^{-2})$, $\frac{\partial u}{\partial r}=-\frac{A}{|x|^2}-\frac{2B_ix^i}{|x|^4}+O(|x|^{-4})$ and $u^4=1+\frac{4A}{|x|}+\frac{6A^2}{|x|^2}+\frac{4B_ix^i}{|x|^3}+O(|x|^{-3})$, which then easily yields (recall $m=2A$, and let $Ac^k=B_k$)

$$\int\limits_{|x|=r}\left[\sum_i x^k\left(g_{ij,i}-g_{ii,j}\right)\nu_e^j d\mu_e-\sum_i\left(g_{ik}\nu_e^i-g_{ii}\nu_e^k\right)d\mu_e\right]=16\pi mc^k+O(1/R).$$

To motivate the definition of c^k, we note that the expansion transforms under translation as $u(y-a) = 1 + \frac{A}{|y|} + \frac{\beta_i y^i}{|y|^3} + O_\infty(|y|^{-3})$, where $\beta_k = B_k + Aa^k$. So, our definition of c^k corresponds to the translation which makes the $|x|^{-2}$-terms in the expansion vanish: $u(y+c) = 1 + \frac{A}{|y|} + O_\infty(|y|^{-3})$.

In summary, then, if we identify the ADM center of mass quantity (2.1) with mc^k, the center of mass c^k transforms appropriately under translation, and in the conformally flat case, it governs the $|x|^{-2}$-term in the expansion, and translation by $(-c^k)$ yields coordinates in which this term of the expansion can be made to vanish.

The mass and center of mass are related to asymptotic symmetries of the space. We note that the linear and constant functions are in the kernel of the adjoint of the linearization of the constraint operator at the Euclidean metric g_e, which is how the mass and center of mass arise in the context of [C1] and [CS]. In the case of conformally flat asymptotics, these quantities arise from conformal Killing fields in the asymptotic region which represent dilation or translation of infinity. To see this, we recall a generalized Pohozaev identity from [S].

Proposition 2.1. *Let (M^n, g) be a compact Riemannian manifold with smooth boundary ∂M. Suppose X is a conformal Killing field on M. Then we have the identity*

$$\int_M X(R(g))dv_g = \frac{2n}{n-2}\int_{\partial M} (Ric_g - \frac{R(g)}{n} g)(X,\nu)d\mu_g,$$

where ν is the outward unit normal to ∂M.

We note that for (M,g) with vanishing scalar curvature, the identity reduces to

$$\int_{\partial M} Ric_g(X,\nu)d\mu_g = 0. \tag{2.3}$$

We consider the case in which M is the exterior of a ball in $\mathbb{R}^3$, with asymptotically flat and conformally flat metric g with mass m, and with vanishing scalar curvature. We now find candidates for the conformal Killing field X. Since dilations and inversions are conformal isometries of the flat metric, the following map is a conformal isometry near infinity, for any $a > 0$:

$$\mathbb{R}^3 \ni x \mapsto \frac{x}{|x|^2} \mapsto \frac{ax}{|x|^2} \mapsto \frac{ax/|x|^2}{a^2/|x|^2} = \frac{x}{a}.$$

By taking the derivative with respect to a at $a=1$, we see that the infinitesimal generator of this family of isometries is $X = -x^i\partial_i$, where the ∂_i form an g_e-orthonormal basis of coordinate vector fields in a conformally flat chart at infinity, so that $g(\partial_i, \partial_j) = u^4\delta_{ij}$. Note that if we lower the index using the metric g, we get $X_i = -u^4x^i$. By design X is a conformal Killing field near infinity, and we could also check this explicitly. Let $Z = -X$. It is straightforward to compute

$$Z_{i;j} = Z_{i,j} - \Gamma^k_{ij}Z_k = 2u^3x^i\partial_j u - 2u^3x^j\partial_i u + \delta_{ij}(u^4 + 2u^3x^k\partial_k u),$$

and this yields the conformal Killing equation

$$Z_{i;j} + Z_{j;i} = 2u^4\delta_{ij}(1 + \frac{2}{u}x^k\partial_k u) = \frac{2}{3}div_g(Z)g_{ij}.$$

If we compute the integral of $Ric_g(X,\nu)$ over a large sphere $S_R=\{|x|=R\}$ in the asymptotically flat region, with the normal ν pointing toward infinity, we get, letting ν_e denote the Euclidean normal,

$$\int_{S_R} Ric_g(X,\nu)d\mu_g = -\int_{S_R} Ric_g(x^j\partial_j,\frac{\nu_e}{u^2})u^4 d\mu_e = -\int_{S_R}\frac{u^2}{|x|}x^jx^kRic_g(\partial_j,\partial_k)d\mu_e.$$

Commputing directly or using the well-known formula for the transformation of Ricci tensor under a conformal transformation [Be], we obtain

$$Ric_g(\partial_j,\partial_k)=R_{jk}=-\frac{2}{u}\partial_{jk}u+\frac{6}{u^2}\partial_j u\,\partial_k u-\frac{2}{u^2}\delta_{jk}|\nabla u|_e^2.$$

It is a simple matter to compute the expansions of the terms in the preceding:

$$\begin{aligned}\partial_j u\,\partial_k u &= \frac{m^2x^jx^k}{4|x|^6}+O(|x|^{-5})\\ \partial_{jk}u &= \big(\frac{3m}{2|x|^5}x^jx^k-\frac{m}{2|x|^3}\delta_{jk}\big)-\frac{3}{|x|^5}(B_jx^k+B_kx^j+B_ix^i\delta_{jk})\\ &\quad+\frac{15}{|x|^7}B_ix^ix^jx^k+O(|x|^{-5}).\end{aligned}$$

Plugging this into the integral, we get

$$\int_{S_R}Ric_g(X,\nu)d\mu_g=\int_{S_R}\frac{2u}{R}x^jx^k\partial_{jk}u\,d\mu_e+O(R^{-1})=8\pi m+O(R^{-1}).$$

If we have that the metric g is globally conformally flat with vanishing scalar curvature, then (2.3), applied to the boundary of large balls, implies that the mass is zero. In this simple case we need not invoke the Positive Mass Theorem to prove the metric is flat, since this of course follows from the maximum principle applied to the harmonic conformal factor u. If g is conformally flat near infinity, then by applying (2.3) to a large annular region, we get an asymptotic conservation statement for the mass integral across the boundaries of the region.

In an entirely similar computation, the center of mass appears if we consider vector fields which generate translations near infinity. That is, for any $y\in\mathbb{R}^3$, we consider the conformal isometry

$$\mathbb{R}^3\ni x\mapsto\frac{x}{|x|^2}+y\mapsto\left|\frac{x}{|x|^2}+y\right|^{-2}\left(\frac{x}{|x|^2}+y\right).$$

Differentiating this at $y=0$ yields the vector fields

$$X_l=|x|^2\partial_l-2x^lx^j\partial_j$$

for $l=1,2,3$. Just as before these are conformal Killing fields near infinity, and the boundary integrals in (2.3) can again be computed by expanding u; in this case, the coefficients of the $|x|^{-2}$-terms in u appear, and as we have seen these coefficients B_i involve the center of mass. Using the above expansion of u, we see

$$\Big(\frac{6}{u^2}\partial_ju\,\partial_ku-\frac{2}{u^2}\delta_{jk}|\nabla u|_e^2\Big)X_l^j\nu^ku^4=-\frac{m^2x^l}{|x|^3}+O(|x|^{-3}).$$

By symmetry, then,

$$\int_{S_R}Ric_g(X_l,\nu)d\mu_g=-\int_{S_R}\frac{2u}{R}X_l^jx^k\partial_{jk}u\,d\mu_e+O(R^{-1}).$$

We plug the above expansion of the Hessian of u into this integral. We note that the leading-order mass terms from the expansion are not absolutely integrable, but the terms have the correct symmetry so that the corresponding surface integral vanishes. It is straightforward to compute the contribution to the surface integral from the other terms in the expansion:

$$\int_{S_R} Ric_g(X_l,\nu)d\mu_g = 32\pi B_l + O(1/R) = 16\pi mc^l + O(1/R).$$

3. The center of mass foliation

In an asymptotically flat chart, the coordinate spheres near infinity are approximate solutions to a constant mean curvature equation. One can perturb these to exact solutions either by an implicit function theorem method [Y] (*cf.* [M]), or by the mean curvature flow [HY]. By uniqueness of the foliation as formulated in [Y], the methods will produce the same foliation near infinity in the case of positive mass, so we will use facts from both analyses below. In the approach of Huisken-Yau, one studies solutions $F^\sigma : \mathbb{S}^2 \times I \to M$ of the flow $\frac{d}{dt}F^\sigma = (h-H)\nu$, with initial condition F_0^σ the standard embedding of the sphere of radius σ centered at the origin. ν is the outward unit normal, H is the inward mean curvature of the surface $F_t^\sigma(\mathbb{S}^2) =: M_t^\sigma$, and h is the integral average of the mean curvature ($\int_{M_t^\sigma}(h-H)d\mu_g = 0$); we also follow convention that A is the second fundamental form, whose components in any local basis are h_{ij}. This normalized mean curvature flow improves the isoperimetric ratio (area decreases while the enclosed volume is fixed). For large σ, the solution exists for all times $t \geq 0$, the surfaces $F_t^\sigma(\mathbb{S}^2)$ converge to surfaces $F_\infty^\sigma(\mathbb{S}^2) := M^\sigma$, and the limiting configuration $\{M^\sigma, \sigma \geq \sigma_0\}$ forms a foliation by stable constant mean curvature spheres. In fact we have the following, which was proven in [HY].

Theorem 3.1. *Consider an asymptotically flat three-manifold (M,g), and suppose in an exterior region the metric can be written $g_{ij} = (1+\frac{m}{2|x|})^4\delta_{ij} + O_4(|x|^{-2})$ with $m > 0$. There is a $\sigma_0 > 0$, and positive constants C_1 and C_2, and a vector $\gamma \in \mathbb{R}^3$ so that for all $\sigma \geq \sigma_0$, the following are true. The initial value problem has a unique smooth solution for all times $t \geq 0$. The surfaces $M_t^\sigma = F_t^\sigma(\mathbb{S}^2)$ converge exponentially fast to a smooth stable hypersurface M^σ, with constant mean curvature H_σ. The radial coordinate $r = |x|$ satisfies $|r-\sigma| \leq C_1$ on M^σ, and $|H_\sigma - \frac{2}{\sigma} + \frac{4m}{\sigma^2}| \leq C_2\sigma^{-3}$. The hypersurfaces M^σ have a joint center of mass vector γ, in the following sense (where $\mathbb{P}$ is the coordinate position vector function, and $d\mu^\sigma$ is the pullback of the Euclidean surface measure on M^σ):*

$$\gamma = \lim_{\sigma\to+\infty} \frac{\int_{M^\sigma} \mathbb{P}\, d\mu_e}{\int_{M^\sigma} d\mu_e} = \lim_{\sigma\to+\infty} \frac{\int_{\mathbb{S}^2} F_\infty^\sigma\, d\mu^\sigma}{\int_{\mathbb{S}^2} d\mu^\sigma}.$$

We note here that the mass is assumed to be positive, but the constraint equations are not imposed, and in particular, no local energy condition is assumed; in fact, Ye only requires that the mass parameter be *nonzero*. The relation of the mass and the effects on the geometry of the three-manifold via the isoperimetric problem arises in the work of Bray [Br] (*cf.* [BM], [CGGK]) on the Penrose Inequality, and more recently on an isoperimetric approach to the definition of mass of isolated systems due to Huisken [H4], [H5], which we only briefly motivate here.

In Euclidean space, the isoperimetric inequality for a closed surface Σ of area A enclosing a volume V can be written $V \leq \frac{A^{3/2}}{6\sqrt{\pi}}$, with equality precisely in the case Σ is a round sphere. We compare this to the Schwarzschild metric of mass $m > 0$, where it is easy to compute the volume $V(A)$ enclosed between the isoperimetric sphere of area A and the horizon, which has the expansion

$$V(A) = \frac{A^{3/2}}{6\pi^{1/2}}\left(1 + \frac{(3\sqrt{\pi})m}{\sqrt{A}} + mO(\frac{1}{A})\right).$$

This illustrates how the mass m measures the deviation of the geometry from that of Euclidean, which is explored in great detail in the recent work of Huisken.

Both Huisken-Yau and Ye argue that the foliation is asymptotically round on approach to infinity; see also the estimates in the more recent work of Metzger [M]. We state following result, which follows directly from [Y]; in particular, notice how the constant mean curvature foliation is produced by solving for a shift τ and a normal perturbation φ of large coordinate spheres.

Theorem 3.2. *Consider an exterior region in an asymptotically flat three-manifold satisying the conditions in the preceding Theorem. Let $0 < \alpha < 1$. There is a $\sigma_0 > 0$ large enough, and a constant $C > 0$ so that for $\sigma \geq \sigma_0$, M^σ is the image of the embedding $\Phi_\rho : \mathbb{S}^2 \ni \omega \mapsto \rho\,(\tau(\rho) + \omega + \varphi(\rho, \omega)\nu(\omega))$, where $\frac{2}{\rho} - \frac{4m}{\rho^2} = H_\sigma$, $\|\varphi(\rho, \cdot)\|_{C^{2,\alpha}(\mathbb{S}^2)} \leq \frac{C}{\rho}$ and $\tau(\rho) \leq \frac{C}{\rho}$. If g_ρ is the induced metric on M^σ, then as $\rho \to +\infty$, $\rho^{-2}\Phi_\rho^*(g_\rho)$ converges in $C^{1,\alpha}$ to the unit round metric on $\mathbb{S}^2$.*

We now state the Main Theorem, which implies in the case where the metric is conformally flat and scalar flat near infinity that the center of mass from the ADM formulation agrees with the geometric center.

Theorem 3.3. *Consider an exterior region in an asymptotically flat manifold, and assume there is an asymptotically flat chart in which $g_{ij} = (1 + \frac{m}{2|x|} + \frac{B_k x^k}{|x|^3})^4\, \delta_{ij} + O_5(|x|^{-3})$ with $m > 0$. Then $\gamma^k = \frac{2B_k}{m} = c^k$ in this chart.*

We require decay through one more derivative than in the theorems above; this is a technical assumption, needed only in (6.9), which is used to prove Proposition 4.2. We stress that we have not imposed any substantial extra decay on the metric, and in fact this assumption holds for metrics which are conformally flat and scalar flat near infinity.

We noted above that translating the coordinates by a^k shifts the ADM center of mass by a^k, and we easily see it also shifts the Huisken-Yau center by the same amount. As we noted above, we can shift coordinates to make the ADM center of mass vector zero, and in these coordinates the expansion of the conformal factor the $|x|^{-2}$-terms in the conformal factor vanish. Thus to prove the Main Theorem it suffices to prove the following case.

Theorem 3.4. *Consider an exterior region in an asymptotically flat manifold, and assume there is an asymptotically flat chart in which $g_{ij} = (1 + \frac{m}{2|x|})^4 \delta_{ij} + O_5(|x|^{-3})$ with $m > 0$. Then $\gamma = 0$ in this chart.*

As we discussed above, spaces which are conformally flat near infinity with vanishing scalar curvature admit such a coordinate chart. The metric g in Theorem 3.4 agrees with Schwarzschild in this chart to one higher order in $|x|^{-1}$ than is generally considered in [HY]. Hence, the radial coordinate spheres $F_0^\sigma(\mathbb{S}^2) = \{|x| = \sigma\}$

in this chart are better approximate solutions; we have arranged this by design by centering them appropriately. It suffices, then, to show that the flow moves them sufficiently little so that the center of mass determined by the foliation is the zero vector (in this chart). We will prove the theorem in the next section. The remainder of this section is devoted to emphasizing some basic facts of the mean curvature flow under the assumptions in Theorem 3.1 which we will need in the proof.

We first note the following lemma, which compares the geometry of the surfaces M_t^σ (always with respect to the metric induced by g) to that of the round sphere of curvature σ^{-2}. Let $\bar{g}_t^\sigma = \sigma^{-2}(F_t^\sigma)^*(g)$ be the metric obtained by pulling back the metric g to the sphere, and rescaling.

Lemma 3.1. *There is a constant $C > 1$ so that for all $t \geq 0$ and all σ large enough, the following inequalites involving the sectional curvature K, diameter d, area A and injectivity radius i of M_t^σ hold:*

$$\begin{aligned} C^{-1}\sigma^{-2} \leq &\quad K(M_t^\sigma) &\leq C\sigma^{-2} \\ C^{-1}\sigma \leq &\quad d(M_t^\sigma) &\leq C\sigma \\ C^{-1}\sigma \leq &\quad i(M_t^\sigma) &\leq C\sigma \\ C^{-1}\sigma^2 \leq &\quad A(M_t^\sigma) &\leq C\sigma^2. \end{aligned}$$

Proof. As in [HY], the principal curvatures λ_i of M_t^σ satisfy $\lambda_i = \frac{1}{\sigma} + O(\sigma^{-2})$, $i = 1, 2$. By the Gauss equation and the decay of the ambient metric to the flat metric, the curvature inequality follows. The area inequality then follows by the Gauss-Bonnet Theorem. The area inequality can also be established by noting that the rate of change of the area is $\frac{dA(M_t^\sigma)}{dt} = -\int_{M_t^\sigma} (h-H)^2 d\mu_g \leq 0$, so the area is decreasing. By the convergence properties of the flow (convergence to a round metric), and the convergence of the initial data on M_0^σ, the area inequality above holds, for σ large enough.

The upper bounds on the diameter and injectivity radius follow from the Bonnet-Myers Theorem. By Klingenberg's Lemma [Do], or from Cheeger's Theorem [Ch] applied to $(\mathbb{S}^2, \bar{g}_t^\sigma)$, we have that the injectivity radius and hence diameter satisfy the required lower bound. ■

The uniform bounds on the geometry yield a uniform Sobolev embedding inequality.

Lemma 3.2. *For all σ sufficently large, for all $t \geq 0$, we have for each $r > 2$ a constant $C = C(r)$ so that for every $u \in W^{1,r}(M_t^\sigma)$, we have (with $q = \frac{r}{2} + 1$)*

$$\|u\|_{C^0} \leq C\left(\sigma^{1-2/r}\|du\|_{L^r(M_t^\sigma)} + \sigma^{-2/q}\|u\|_{L^r(M_t^\sigma)}^{1/q}\|u\|_{L^2(M_t^\sigma)}^{1-1/q}\right).$$

Proof. Let $r > 2$, and let $q = \frac{r}{2} + 1$, so that $2 < q < r$. By the uniform injectivity estimate from below, and the uniform curvature estimate from above, there is a constant $C = C(q)$ so that for all $u_\sigma \in W^{1,q}(\mathbb{S}^2, \bar{g}_t^\sigma)$, $\|u_\sigma\|_{C^0} \leq C\|u_\sigma\|_{W^{1,q}(\bar{g}_t^\sigma)}$ (*cf.* [A], pp. 45-46). By scaling, with $u_\sigma = u \circ F_t^\sigma$, this is just

$$\|u\|_{C^0} \leq C\sigma^{-2/q}\big(\sigma\|du\|_{L^q(M_t^\sigma)} + \|u\|_{L^q(M_t^\sigma)}\big).$$

We apply the interpolation inequality $\|v\|_{L^q} \leq \|v\|_{L^r}^{1/q}\|v\|_{L^2}^{(1-1/q)}$ (which follows from Hölder's inequality [GT]) to obtain

$$\|u\|_{C^0} \leq C\sigma^{-2/q}\big(\sigma\|du\|_{L^q(M_t^\sigma)} + \|u\|_{L^r(M_t^\sigma)}^{1/q}\|u\|_{L^2(M_t^\sigma)}^{1-1/q}\big). \tag{3.1}$$

Applying the Hölder inequality and using the area bound from the preceding lemma, we obtain $\sigma\|du\|_{L^q(M_t^\sigma)} \leq \sigma C\sigma^{2/q-2/r}\|du\|_{L^r(M_t^\sigma)}$, which finishes the proof. ∎

We now apply Lemma 3.1 to get a bound on $|H-h|$.

Lemma 3.3. *There is a constant $C>0$ so that for all $t\geq 0$ and all σ large enough, $|\nabla H|^2 \leq C\sigma^{-8}$. Therefore, $|H-h|\leq C\sigma^{-3}$.*

Proof. The estimate on the gradient follows from [HY] (*cf.* Corollary 3.11 and Proposition 3.12 therein). The estimate on $|H-h|$ now follows by integration of ∇H along geodesics, using the fact that the diameter of M_t^σ is bounded by $C\sigma$, for some constant C, by Lemma 3.1. ∎

We apply the preceding result to establish that a key differential inequality from [HY] holds for all $t\geq 0$.

Proposition 3.4. *There is a $\sigma_0>0$ and an $\epsilon>0$ so that for all $\sigma\geq\sigma_0$ and for all $t\geq 0$,*

$$\frac{d}{dt}\int_{M_t^\sigma}(H-h)^2\, d\mu_g \leq -\frac{12-2\epsilon}{\sigma^3}\int_{M_t^\sigma}(H-h)^2\, d\mu_g. \tag{3.2}$$

Thus

$$\int_{M_t^\sigma}(H-h)^2\, d\mu_g \leq e^{-\frac{(12-2\epsilon)t}{\sigma^3}}\int_{M_0^\sigma}(H-h)^2\, d\mu_g. \tag{3.3}$$

Proof. We follow Huisken-Yau. Indeed, we start with the evolution equation [HY]

$$\begin{aligned}\frac{d}{dt}\int_{M_t^\sigma}(H-h)^2\, d\mu_g &= 2\int_{M_t^\sigma}(H-h)\Delta(H-h)\, d\mu_g \\ &\quad + \int_{M_t^\sigma}[2(H-h)^2(|A|^2+Ric_g(\nu,\nu))-(H-h)^3H]\, d\mu_g.\end{aligned}$$

As in [HY], the lowest eigenvalue λ_1 of the Laplace operator on the hypersurface M_t^σ satisfies $\lambda_1 \geq \frac{2}{\sigma^2}-\frac{4m}{\sigma^3}-C\sigma^{-4}$, while the curvature terms satisfy $|A|^2+Ric_g(\nu,\nu)\leq \frac{2}{\sigma^2}-\frac{10m}{\sigma^3}+C\sigma^{-4}$. Thus after integrating by parts, and using the fact that $\int_{M_t}(H-h)\, d\mu_g=0$, we have for σ large enough,

$$\frac{d}{dt}\int_{M_t^\sigma}(H-h)^2\, d\mu_g \leq -\frac{(12-\epsilon)m}{\sigma^3}\int_{M_t^\sigma}(H-h)^2\, d\mu_g - \int_{M_t^\sigma}(H-h)^3H\, d\mu_g.$$

Now by the preceding lemma, for all $t\geq 0$ and σ large enough, $|H(H-h)|\leq C\sigma^{-4}\leq \epsilon m\sigma^{-3}$. The result follows. ∎

In the next section, we will use interpolation and the bounds on the derivatives of the curvature to turn the L^2-bound coming from this inequality into a pointwise bound.

3.1. **Remarks.** The bound on $|H-h| = O(\sigma^{-3})$ is of course intimately tied to the behavior of the flow, and hence the center of mass. In fact we now remark that the σ^{-3}-bound derives from a uniform bound in space and time of the geometric centers of the evolving surfaces. Indeed, by [HY], for all $t \geq 0$ and all σ large enough, there is a vector a (depending on t and σ) so that $|F_t^\sigma - a - r_0\nu_e| \leq C\sigma^{-1}$, for some r_0 satisfying $|r_0 - \sigma| \leq C$; moreover,

$$|h - H| \leq \frac{6m|a|_e}{r_0^3} + O(\sigma^{-3}) \tag{3.4}$$

(*cf.* Proposition 2.2, Theorem 3.3, Proposition 3.4 in [HY]). Since $\int_{M_t^\sigma} \nu_e \, d\mu_e = 0$ by Stokes' Theorem, and since $|d\mu_g - d\mu_e| \leq C\sigma^{-1} d\mu_g$ on M_t^σ, we have that a is equal to $\alpha + O(1)$, where $\alpha = \alpha(t,\sigma) = \frac{\int_{M_t^\sigma} \mathbb{P}\, d\mu_g}{\int_{M_t^\sigma} d\mu_g}$. We note that $\alpha = O(\sigma^{-1})$ at $t = 0$, since the induced surface measure on the coordinate spheres is round to $O(\sigma^{-2})d\mu_g$. Now we compute the time derivative of α, getting

$$\partial_t \alpha = \frac{1}{\int_{M_t^\sigma} d\mu_g} \left[\int_{M_t^\sigma} \partial_t \mathbb{P}\, d\mu_g + \int_{M_t^\sigma} (\mathbb{P} - \alpha)\partial_t(d\mu_g) \right].$$

A routine calculation shows $\partial_t(d\mu_g) = H(h - H)d\mu_g$. Plugging this along with the flow equation into the preceding, we obtain

$$\partial_t \alpha = \frac{\int_{M_t^\sigma} (h - H)[\nu + H(\mathbb{P} - \alpha)]d\mu_g}{\int_{M_t^\sigma} d\mu_g}.$$

By Cauchy-Schwarz, we thus conclude that $|\partial_t \alpha| \leq C\sigma^{-1}\|h - H\|_{L^2(M_t^\sigma)}$. Using $|h - H| = O(\sigma^{-3})$ at $t = 0$, and using (3.3), we have by integration that α, and hence a is $O(1)$. Thus by (3.4), $|h-H| = O(\sigma^{-3})$ uniformly in σ large and $t \geq 0$. ∎

We let g_0 be the metric on a round unit sphere. One may obtain the geometric facts above by proving that the metrics $\bar{g}_t^\sigma$ are uniformly equivalent to g_0 for large σ and all $t \geq 0$. One is led to conjecture this, since these metrics are obtained by rescaling in space and time the mean curvature flows which define M_t^σ. Indeed let $\Psi_\rho : \mathbb{R}^3 \setminus \{|x| \leq 1\} \to \mathbb{R}^3 \setminus \{|x| \leq \rho\}$, where $\Psi_\rho(x) = \rho x$. Then let $\bar{g}_\rho = \rho^{-2}\Psi_\rho^*(g)$. If $F = F_t^\rho$ denotes the solution of $\frac{\partial F}{\partial t} = (h - H)\nu$ with initial data induced from g on $\{|x| = \rho\}$, we let $\tilde{F}$ denote the pullback map: $F = \Psi_\rho \circ \tilde{F}$. We then note that

$$\frac{\partial F}{\partial t} = (\Psi_\rho)_* \left(\frac{\partial \tilde{F}}{\partial t} \right) = (h - H)\nu = (\Psi_\rho)_*(\rho^{-2}(\tilde{h} - \tilde{H})\tilde{\nu}).$$

Thus upon rescaling the time coordinate to $\tau = \rho^{-2}t$, we have $\frac{\partial \tilde{F}}{\partial \tau} = (\tilde{h} - \tilde{H})\tilde{\nu}$. Thus the map $\tilde{F}$ satisfies the mean curvature equation with time variable τ, as embeddings into the metric $\bar{g}_\rho$. As $\bar{g}_\rho$ converges (smoothly uniformly on compact subsets) to the Euclidean metric, the initial data for $\tilde{F}$ converges to the unit sphere in Euclidean space, as $\rho \to +\infty$. The flows converge to metrics near a standard unit sphere, *cf.* Theorem 3.2. It seems reasonable that along the flows the metrics stay uniformly equivalent. In fact, Huisken and Yau show that the under the flow the geometry is controlled (*cf.* Theorem 3.3 in [HY]). To be precise, the inequality $|F_t^\sigma - a - r_0\nu_e| \le C\sigma^{-1}$ is proved by showing in adapted coordinates that $|\frac{\partial}{\partial y^i}(\nu_e - r_0^{-1}F_t^\sigma)| = O(\sigma^{-3})$, so that the equivalence of the metrics can easily be derived from this, the bound on curvatures, and the Gauss-Weingarten relation $\frac{\partial \nu_e}{\partial y^i} = h_{ij}^e g_e^{jk}\frac{\partial F_t^\sigma}{\partial y^k}$. Thus under the flow, the metrics induced by g on M_t^σ are uniformly comparable to the metrics on round spheres of curvature σ^{-2}. This gives Lemma 3.1. We can also get a uniform Sobolev bound as above, or as follows.

Proposition 3.5. *Let $q > 2$. For all $t \ge 0$ and all σ large enough, there is a constant $C = C(q)$ for the Sobolev embedding inequality for functions u with $\int_{M_t^\sigma} u\, d\mu_g = 0$ holds:*

$$\|u\|_{C^0(M_t^\sigma)} \le C\sigma^{1-2/q}\|du\|_{L^q(M_t^\sigma)}. \tag{3.5}$$

Proof. We note the inequality $\|u\|_{W^{1,q}(\mathbb{S}^2)} \le C\|du\|_{L^q(\mathbb{S}^2)}$ on the standard unit sphere, for functions of mean zero. Indeed, if there were a sequence of mean-zero functions u_i of unit $W^{1,q}(\mathbb{S}^2)$-norm, so that du_i converges to zero in $L^q(\mathbb{S}^2)$, then by compactness of the embedding $W^{1,q}(\mathbb{S}^2) \hookrightarrow L^q(\mathbb{S}^2)$, we can actually assume without loss of generality that the sequence converges in $W^{1,q}(\mathbb{S}^2)$ to a limit u, with $du = 0$. Moreover, $\|u\|_{W^{1,q}(\mathbb{S}^2)} = 1$. Since $du = 0$, u is constant; but since u must be orthogonal to the constants (since the u_i are), we have that $u = 0$. This is a contradiction. By scaling, then, the estimate holds in case M_t^σ are Euclidean spheres of radius σ. By uniform metric equivalence, then, we can choose a constant C which will suffice for the entire family of surfaces. ■

4. Proof of the main theorem

To prove Theorem 3.4, we need to estimate carefully how much F_t^σ varies from F_0^σ, for $0 < t \le +\infty$, and to keep track of the variation in the induced surface measure on M_t^σ. Both of these quantities evolve by an equation involving $(H-h)$, as $\partial_t F_t^\sigma = (h-H)\nu$, and $\partial_t(d\mu_g) = H(H-h)d\mu_g$. For $t \ge 0$, let $m(t,\sigma) = \max\limits_{M_t^\sigma}|H-h|$. The goal, then, is to show $m(t,\sigma)$ decays sufficiently fast in space and time, under the hypotheses of Theorem 3.4 (under which we work from now on). For this, we have the following proposition.

Proposition 4.1. *There is a $C > 0$ so that for all $\sigma \ge \sigma_0$, and for all $t \ge 0$,*

$$m(t,\sigma) \le C\sigma^{-7/2}e^{-5\sigma^{-3}t/2}. \tag{4.1}$$

In the course of the proof of this proposition, we will use the following bound on the Hessian of H.

Proposition 4.2. *There is a $C > 0$ so that for all $t > 0$ and for all σ large enough,* $|\nabla^2 H|^2 \le C\sigma^{-10}$.

Proof. See the Appendix. ■

Proof of Proposition 4.1. We begin by estimating $(H-h)$ at $t = 0$. Recall that we are working in a coordinate system where the metric g decays to Schwarzschild g^S as $g_{ij}(x) - g_{ij}^S(x) = O_5(|x|^{-3})$. Thus, the difference between the connections satisfies $\Gamma - \Gamma^S = O(|x|^{-4})$, and the differences of the unit normal vectors to the radial sphere $\{|x| = \sigma\}$ satisfies $\nu - \nu^S = O(\sigma^{-3})$ (measured in either metric); this latter fact follows by writing $\nu^S = c\nu + c^1 e_1 + c^2 e_2$, where $\{e_1, e_2\}$ form a g-orthonormal basis for the tangent plane, and noting that $c^i = O(\sigma^{-3})$, so that $c = 1 + O(\sigma^{-3})$. We thus have (in an adapted coordinate system, *e.g.* Euclidean spherical coordinates)

$$A_{ij} - A^S_{ij} = (\Gamma - \Gamma^S)^k_{ij}\nu^l g_{kl} + (\Gamma^S)^k_{ij}(\nu - \nu^S)^l g_{kl} + (\Gamma^S)^k_{ij}(\nu^S)^l (g_{kl} - g^S_{kl}). \tag{4.2}$$

Thus $(H - h) = O(\sigma^{-4})$ at $t = 0$, and so by (3.3), we have

$$\int_{M^\sigma_t} (H - h)^2 \, d\mu_g \le C\sigma^{-6} e^{-(12-2\epsilon)t/\sigma^3}. \tag{4.3}$$

To turn this into a pointwise bound, we use the following interpolation inequality from Hamilton [Ha]: if T is any tensor on a closed surface M, and for any $2 < q < 4$, then (independent of g and the connection) for $\frac{1}{p} + \frac{1}{2} = \frac{2}{q}$, we have

$$\left(\int_M |dT|^q \, d\mu_g\right)^{2/q} \le q \left(\int_M |\nabla^2 T|^p \, d\mu_g\right)^{1/p} \cdot \left(\int_M |T|^2 \, d\mu_g\right)^{1/2}.$$

If we apply this to M^σ_t and $T = H - h$, and use the estimate from Proposition 4.2 on the Hessian of the mean curvature, and the L^2-bound on $(H - h)$ from (4.3), we obtain

$$\begin{aligned}
\sigma^{1-2/q}\|d(H-h)\|_{L^q(M^\sigma_t)} &\le q\sigma^{1-2/q}\|\nabla^2(H-h)\|^{1/2}_{L^p(M^\sigma_t)}\|H-h\|^{1/2}_{L^2(M^\sigma_t)} \\
&\le C\sigma^{1-2/q} \cdot (\sigma^{-5}\sigma^{2/p})^{1/2} \cdot (\sigma^{-6}e^{-10\sigma^{-3}t})^{1/4} \\
&= C\sigma^{-7/2}e^{-5\sigma^{-3}t/2}.
\end{aligned}$$

This bounds the first term on the right side of (3.1), and the other term can be bounded in a similar fashion: for $q = \frac{r}{2} + 1$ and $r > 2$,

$$\begin{aligned}
\sigma^{-\frac{2}{q}}\|H-h\|^{\frac{1}{q}}_{L^r(M^\sigma_t)}\|H-h\|^{1-\frac{1}{q}}_{L^2(M^\sigma_t)} &\le C\sigma^{-\frac{2}{q}} \cdot (\sigma^{-3}\sigma^{\frac{2}{r}})^{\frac{1}{q}} \cdot (\sigma^{-6}e^{-10\sigma^{-3}t})^{\frac{1}{2}(1-\frac{1}{q})} \\
&= C\sigma^{-3-\frac{2}{q}(1-\frac{1}{r})} \cdot (e^{-10\sigma^{-3}t})^{\frac{1}{2}(1-\frac{1}{q})}.
\end{aligned}$$

By choosing $r \in (2, 4]$, we can show that in fact the exponent of σ in the preceding at most $-\frac{7}{2}$. From the Sobolev inequality, this yields the desired result: $|H - h| \le C\sigma^{-7/2}e^{-5\sigma^{-3}t/2}$. ■

By integrating the flow equations $\partial_t F^\sigma_t = (h - H)\nu$, and $\partial_t(d\mu_g) = H(H-h)d\mu_g$, we immediately obtain the following decay estimates.

Proposition 4.3.

$$\begin{aligned}
|F^\sigma_\infty - F^\sigma_0| &\le C\sigma^{-1/2} \\
|(F^\sigma_\infty)^*(d\mu_g) - (F^\sigma_0)^*(d\mu_g)| &\le C\sigma^{-3/2}|(F^\sigma_0)^*(d\mu_g)|.
\end{aligned}$$

We remark that the general bound $H - h = O(\sigma^{-3})$ results in a slightly weaker L^2-bound, so that the exponential decay of $H - h$ under the flow becomes $|H - h| \le C\sigma^{-3}e^{-5\sigma^{-3}t/2}$. As expected, $\int\limits_0^{+\infty} |H - h| \, dt = O(1)$ in the general case.

Proof of Theorem 3.4. We use the estimate in the preceding proposition, and the form of the metric g to conclude that the ratio of $d\mu^\sigma = (F^\sigma_\infty)^*(d\mu_e)$ to a round metric is $(1+O(\sigma^{-3/2}))$. Thus the metric is symmetric enough so that

$$\frac{\int_{M^\sigma} \mathbb{P}\, d\mu_e}{\int_{M^\sigma} d\mu_e} = \frac{\int_{\mathbb{S}^2} F^\sigma_\infty\, d\mu^\sigma}{\int_{\mathbb{S}^2} d\mu^\sigma} = O(\sigma^{-1/2}).$$

This obviously yields the theorem. ■

REMARK. Another way to complete the proof is to consider the map $G^\sigma_t = F^\sigma_t + (F^\sigma_t)^-$, $0 \le t \le +\infty$, where for any function f on $\mathbb{S}^2$, we define $f^-(p) = f(-p)$. The quantity G^σ_t measures the failure of equivariance of the antipodal maps on the domain and range and the flow. By design, G^σ_0 vanishes identically, and we can estimate its evolution using the flow equation $\partial_t G^\sigma_t = (h-H)\nu + (h-H)^-\nu^-$. The symmetry estimates of the surface measures along with the decay of $m(t,\sigma)$ yield

$$2\gamma = \lim_{\sigma\to+\infty} \frac{\int_{\mathbb{S}^2} G^\sigma_\infty\, d\mu^\sigma}{\int_{\mathbb{S}^2} d\mu^\sigma} = 0. \quad ■$$

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6. Appendix

In this section we prove the bound on $\nabla^2 H$ stated in Proposition 4.2. We are assuming the hypothesis of Theorem 3.4 holds. The proof is an adaptation of the maximum principle arguments used to obtain bounds on the trace-free part of the second fundamental form and its first derivative in [HY].

To begin, we recall some basic facts that will be used throughout the argument. First, a simple computation shows that the Ricci curvature of the Schwarzschild metric $g^S_{ij} = (1+\frac{m}{2|x|})^4\delta_{ij}$ is given by

$$R^S_{ij}(x) = \frac{m}{|x|^3}\left(1+\frac{m}{2|x|}\right)^{-2}\left(\delta_{ij} - 3\frac{x^i x^j}{|x|^2}\right).$$

If $g = g_S + O_5(|x|^{-2})$, then $|\nabla^k_g Ric_g - \nabla^k_{g_S} Ric_{g_S}| \le C|x|^{-4-k}$, for $k = 0,1,2,3$. Moreover, $|Ric_g(\nu, e_0)| \le C\sigma^{-4}$, where ν is the unit normal vector and e_0 is a unit tangent vector to M^σ_t; this follows directly from the form of the Ricci curvature given above, and the existence of an approximate center, as discussed following Proposition 3.4. This estimate implies $|\nabla A|^2 \le C\sigma^{-8}$, as in [HY] 3.11-3.12.

We will use the following simple equation for tensors T, where $\langle\ ,\rangle$ is the inner product on tensors induced by g:

$$\Delta(|\nabla^m T|^2) = 2|\nabla^{m+1}T|^2 + 2\langle\Delta(\nabla^m T), \nabla^m T\rangle. \tag{6.1}$$

We also record here the following evolution equations [HY]:

$$\partial_t g_{ij} = 2(h-H)h_{ij} \tag{6.2}$$
$$\partial_t H = \Delta H + (H-h)(|A|^2 + Ric_g(\nu,\nu)). \tag{6.3}$$

If T is a tensor, we let $*T$ be a tensor obtained from T by contractions and, in case T is the restriction of an ambient tensor, possibly by interior product with the hypersurface unit normal. We will furthermore let $S*T$ be a linear combination of contractions of the tensor $S\otimes T$ (or of $S\otimes *T$) and similarly for $S*T*U$, and so on. In the derivation below, ∇ is the connection on a hypersurface evolving under the flow, and $\overline{\nabla}$ is the ambient connection, and similarly for the curvature tensors R and $\bar{R}$. We note that if we take an ambient tensor T and perform an interior product with the unit normal to a hypersurface to obtain a tensor $*T$, then the following identity for covariant differentiation along the hypersurface follows from the definition of the second fundamental form: $(\overline{\nabla}_X T) - \nabla_X(*T) = A(X)*T$.

Lemma 6.1. *There is a constant C so that for all $t\geq 0$ and all σ sufficiently large, $|\nabla^2 A|^2 \leq C\sigma^{-8}$.*

Assuming this lemma for the moment, we now prove Proposition 4.2.

Proof of Proposition 4.2. Let $f = H-h$. We compute $\partial_t(|\nabla f|^2)$ using the facts that $(\partial_t f)_{,i} = \partial_t(f_{,i})$ and $\partial_t g^{ij} = 2fh^{ij}$:

$$\begin{aligned}
\partial_t(|\nabla f|^2) &= 2fh^{ij}f_{,i}f_{,j} + 2g^{ij}\partial_t(f_{,i})f_{,j}\\
&= fA*\nabla f*\nabla f + 2g^{ij}\partial_t H_{,i}f_{,j}\\
&= fA*\nabla f*\nabla f + 2g^{ij}(\Delta f + f(|A|^2+Ric_g(\nu,\nu))_{,i}f_{,j}\\
&= fA*\nabla f*\nabla f + 2\langle\nabla(\Delta f),\nabla f\rangle + (|A|^2+Ric_g(\nu,\nu))\nabla f*\nabla f\\
&\quad + f\nabla f*A*\nabla A + f\nabla f*\nabla(Ric_g(\nu,\nu)).
\end{aligned}$$

Using (6.1) along with the following consequence of the Ricci identity,

$$\langle\nabla(\Delta f),\nabla f\rangle = \langle\Delta(\nabla f),\nabla f\rangle + R*\nabla f*\nabla f,$$

and the bounds $f = O(\sigma^{-3})$, $|A| = O(\sigma^{-1})$, and $|\nabla A| = O(\sigma^{-4})$, we obtain

$$\partial_t(|\nabla f|^2) \leq \Delta(|\nabla f|^2) - 2|\nabla^2 f|^2 + c_1|\nabla f|^2\sigma^{-2} + c_2|\nabla f|\sigma^{-7}. \tag{6.4}$$

To compute $\partial_t(|\nabla^2 f|^2)$, we will need the following identity, which follows from $\partial_t\Gamma^m_{ij} = \frac{1}{2}g^{mp}\big((\partial_t g)_{ip;j} + (\partial_t g)_{pj;i} - (\partial_t g)_{ij;p}\big)$:

$$\partial_t(f_{;ij}) = (\partial_t f)_{;ij} + (\partial_t\Gamma^m_{ij})f_{,m} = (\partial_t f)_{;ij} + f\nabla f*\nabla A + \nabla f*\nabla f*A.$$

We will also need the following, which is again consequence of the Ricci identity:

$$(\Delta f)_{;ij} = \Delta(f_{;ij}) + R*\nabla^2 f + \nabla R*\nabla f.$$

Note that we can apply the Gauss equation to get the estimate $\nabla R = O(\sigma^{-4})$.

We now compute (using Lemma 6.1 in the last step)

$$\begin{aligned}
\partial_t(|\nabla^2 f|^2) &= \partial_t(g^{ik}g^{jl}f_{;ij}f_{;kl}) \\
&= 4fh^{ik}g^{jl}f_{;ij}f_{;kl} + 2g^{ik}g^{jl}\big((\partial_t f)_{;ij}f_{;kl} + (\partial_t\Gamma^m_{ij})f_{,m}f_{;kl}\big) \\
&= 4fh^{ik}g^{jl}f_{;ij}f_{;kl} \\
&\quad +2g^{ik}g^{jl}\big[(\Delta f + f(|A|^2 + Ric_g(\nu,\nu))_{;ij}f_{;kl} + (\partial_t\Gamma^m_{ij})f_{,m}f_{;kl})\big] \\
&= \Delta(|\nabla^2 f|^2) - 2|\nabla^3 f|^2 + fA*\nabla^2 f*\nabla^2 f + R*\nabla^2 f*\nabla^2 f \\
&\quad +\nabla R*\nabla f*\nabla^2 f + (|A|^2 + Ric_g(\nu,\nu))\nabla^2 f*\nabla^2 f \\
&\quad +\nabla f*\nabla^2 f*\nabla(|A|^2 + Ric_g(\nu,\nu)) + f\nabla^2 f*\nabla^2(|A|^2 + Ric_g(\nu,\nu)) \\
&\quad +f\nabla A*\nabla f*\nabla^2 f + A*\nabla f*\nabla f*\nabla^2 f \\
&\leq \Delta(|\nabla^2 f|^2) - 2|\nabla^3 f|^2 + c_3\sigma^{-2}|\nabla^2 f|^2 + c_4\sigma^{-8}|\nabla^2 f|.
\end{aligned}$$

Using (6.4) we see

$$\partial_t(|\nabla^2 f|^2 + c_3\sigma^{-2}|\nabla f|^2) \leq \Delta(|\nabla^2 f|^2 + c_3\sigma^{-2}|\nabla f|^2) - c_3\sigma^{-2}|\nabla^2 f|^2 + c_5\sigma^{-12}.$$

We note that we used Lemma 6.1 to estimate $\sigma^{-8}|\nabla^2 f| = O(\sigma^{-12})$, but this is not essential to the following argument. We now pick $D > 0$ so that $(|\nabla^2 f|^2 + c_3\sigma^{-2}|\nabla f|^2) < D\sigma^{-10}$ for $t = 0$ and all σ sufficiently large; the existence of D follows by differentiation of (4.2). If there is a first time $t_0 > 0$ at which $(|\nabla^2 f|^2 + c_3\sigma^{-2}|\nabla f|^2)_{|(y_0,t_0)} = D\sigma^{-10}$, then by choosing D sufficiently large and plugging into the preceding inequality, we get a contradiction to the maximum principle. ■

The proof of the Lemma is entirely similar, but we sketch it for completeness.

Proof of Lemma 6.1. We note first that $\partial_t(h_{ij;k}) = (\partial_t h_{ij})_{;k} + A*\partial_t\Gamma = (\partial_t h_{ij})_{;k} + A*A*\nabla A$. We also recall from [HY] the following identity:

(6.5) $$\partial_t h_{ij} = \Delta h_{ij} + A*A*A + A*\bar{R} + (*\overline{\nabla}\bar{R}).$$

We first compute $\partial_t(|\nabla A|^2)$:

$$\begin{aligned}
\partial_t(|\nabla A|^2) &= \partial_t(g^{ip}g^{jq}g^{kr}h_{ij;k}h_{pq;r}) = 2g^{ip}g^{jq}g^{kr}\partial_t(h_{ij;k})h_{pq;r} + fA*\nabla A*\nabla A \\
&= 2g^{ip}g^{jq}g^{kr}(\partial_t h_{ij})_{;k}h_{pq;r} + A*A*\nabla A*\nabla A \\
&= 2g^{ip}g^{jq}g^{kr}(\Delta h_{ij})_{;k}h_{pq;r} + A*A*\nabla A*\nabla A + \nabla A*\nabla A*\bar{R} \\
&\quad +A*\nabla A*\nabla\bar{R} + \nabla A*\nabla(*\overline{\nabla}\bar{R}).
\end{aligned}$$

We use the Ricci identity to get

(6.6) $$h_{ij;mnk} = h_{ij;mkn} + \nabla A*R = h_{ij;kmn} + \nabla A*R + A*\nabla R.$$

Putting this into the preceding, we obtain

$$\begin{aligned}
\partial_t(|\nabla A|^2) &= \Delta(|\nabla A|^2) - 2|\nabla^2 A|^2 + (\nabla A*R + A*\nabla R)*\nabla A \\
&\quad +A*A*\nabla A*\nabla A + \nabla A*\nabla A*\bar{R} \\
&\quad +A*\nabla A*\overline{\nabla}\bar{R} + A*\nabla A*A*\bar{R} + \nabla A*\overline{\nabla}^2\bar{R} \qquad (6.7) \\
&\leq \Delta(|\nabla A|^2) - 2|\nabla^2 A|^2 + C\sigma^{-9}. \qquad (6.8)
\end{aligned}$$

Next, we compute $\partial_t(|\nabla^2 A|^2)$. Proceeding as above, we have

$$\partial_t(|\nabla^2 A|^2) = 2\langle\partial_t(\nabla^2 A), \nabla^2 A\rangle + fA*\nabla^2 A*\nabla^2 A.$$

To commute the time derivative past the covariant derivative, we proceed as above, using Equation (6.5) in the last line:

$$\begin{aligned}
\partial_t(h_{ij;kl}) &= (\partial_t(h_{ij;k}))_{;l} + \nabla A * \partial_t\Gamma \\
&= [(\partial_t h_{ij})_{;k} + A * \partial_t\Gamma]_{;l} + \nabla A * \partial_t\Gamma \\
&= (\partial_t h_{ij})_{;kl} + A * \nabla A * \nabla A + A * A * \nabla^2 A \\
&= [\Delta h_{ij} + A * A * A + A * \bar{R} + (*\overline{\nabla}\bar{R})]_{;kl} \\
&\quad + A * \nabla A * \nabla A + A * A * \nabla^2 A.
\end{aligned}$$

In a similar manner as above, we obtain

$$h_{ij;mnpq} = h_{ij;pqmn} + \nabla^2 A * R + \nabla A * \nabla R + A * \nabla^2 R,$$

which yields

$$\nabla^2(\Delta A) = \Delta(\nabla^2 A) + \nabla^2 A * R + \nabla A * \nabla R + A * \nabla^2 R.$$

Putting this all together we obtain (using the Gauss equation again to get $\nabla^2 R = O(\sigma^{-5}) + A * \nabla^2 A$)

$$\begin{aligned}
\partial_t(|\nabla^2 A|^2) &= \Delta(|\nabla^2 A|^2) - 2|\nabla^3 A|^2 + \nabla^2 A * \big[A * A * \nabla^2 A + A * \nabla A * \nabla A \\
&\quad \nabla^2 A * R + \nabla A * \nabla R + A * \nabla^2 R + \nabla^2(A * \bar{R}) + \nabla^2(*\overline{\nabla}\bar{R})\big] \\
&\leq \Delta(|\nabla^2 A|^2) + C_1\sigma^{-2}|\nabla^2 A|^2 + C_2\sigma^{-6}|\nabla^2 A|. \qquad (6.9)
\end{aligned}$$

Using (6.8), we see

$$\begin{aligned}
\partial_t(|\nabla^2 A|^2 + C_1\sigma^{-2}|\nabla A|^2) &\leq \Delta(|\nabla^2 A|^2 + C_1\sigma^{-2}|\nabla A|^2) - C_1|\nabla^2 A|^2\sigma^{-2} \\
&\quad + C_2|\nabla^2 A|\sigma^{-6} + CC_1\sigma^{-11}.
\end{aligned}$$

It is easy to verify using (4.2) that there is a $D > 0$ so that at $t = 0$, $(|\nabla^2 A|^2 + C_1\sigma^{-2}|\nabla A|^2) < D\sigma^{-8}$. By choosing D large enough, the maximum principle proves that the inequality persists for all $t > 0$. ■

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