FOUR-MANIFOLDS AND THEIR SYMMETRY GROUPS

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We begin with a brief historical account of the classification of four-dimensional manifolds – up to homotopy; up to homeomorphism; and up to diffeomorphism, with emphasis on the simply-connected case, and a reminder of how the groundbreaking work of both Freedman and Donaldson in the 1980's revealed deep differences between the topological and smooth categories. We will describe the algebra (namely, the intersection form) which classifies the topological case almost completely.

We then consider *group actions* on these manifolds. What groups can arise as symmetries of simply-connected four-manifolds, and what can these symmetries look like? When "large" finite symmetry groups act, the singular strata form a network of intersecting surfaces, and the structure of this network places strong restrictions on the possible groups.

The main algebraic tool which detects the relationships between these surfaces and the ambient manifold is the Borel equivariant cohomology functor: $H^*_G(X) := H^*(EG \times_G X)$. The functor yields a module over the cohomology ring of G, and can be analyzed using the spectral sequence of the Borel fibration $X \to EG \times_G X \to BG$. We will describe the surfaces and outline the proof of the following

Theorem 1. *Let M be a closed, simply-connected four-manifold with $b_2(M) \geq 3$. If a compact Lie group (possibly finite!) G acts effectively, locally linearly and homologically trivially on M, then G is isomorphic to a subgroup of $S^1 \times S^1$.*

A particularly nice picture arises in the case of the groups $G = \mathbb{Z}_p \times \mathbb{Z}_p$. When a manifold has $\mathbb{Z}_p \times \mathbb{Z}_p$ symmetry, the surfaces in the singular set collectively span $H_2(M)$, and the manifold decomposes (via connected sums) into a well-understood set of "building blocks". It is thus possible to classify the actions themselves modulo actions on a single part homeomorphic to S^4. We will describe the actions, and if time permits, discuss the remaining ambiguities about the S^4 summands.